

Degenerate elliptic operators in L_p -spaces with complex $W^{2,\infty}$ -coefficients

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Abstract

Let $c_{kl} \in W^{2,\infty}(\mathbb{R}^d, \mathbb{C})$ for all $k, l \in \{1, \dots, d\}$. We consider the divergence form operator $A = -\sum_{k,l=1}^d \partial_l(c_{kl} \partial_k)$ in $L_2(\mathbb{R}^d)$ when the coefficient matrix satisfies $(C(x)\xi, \xi) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, where Σ_θ be the sector with vertex 0 and semi-angle θ in the complex plane. We show that for all p in a suitable interval the contraction semigroup generated by $-A$ extends consistently to a contraction semigroup on $L_p(\mathbb{R}^d)$. For those values of p we present a condition on the coefficients such that the space $C_c^\infty(\mathbb{R}^d)$ of test functions is a core for the generator on $L_p(\mathbb{R}^d)$. We also examine the operator A separately in the more special Hilbert space $L_2(\mathbb{R}^d)$ setting and provide more sufficient conditions such that $C_c^\infty(\mathbb{R}^d)$ is a core.

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1 Introduction

It has been known for a long time that the space of test functions $C_c^\infty(\mathbb{R}^d)$ is always a core for a strongly elliptic second-order differential operator in divergence form with Lipschitz continuous coefficients. Nevertheless if the operator is merely degenerate elliptic, the situation is very different and it is much more difficult to prove the same type of results. In fact $C_c^\infty(\mathbb{R}^d)$ is no longer a core in general. Some sharp results are available in one dimension which provide characterisations for when $C_c^\infty(\mathbb{R})$ constitutes a core, such as [CMP98, Theorem 3.5], [DE15, Theorem 1.5] and [Do16, Theorem 3.3]. However the techniques used to prove these characterisations are intrinsically available in one dimension only. Up to now extensions of the characterisations to higher dimensions remain widely open problems. On the other hand, some positive results in higher dimensions are also available. Wong-Dzung in [WD83] proved that if a degenerate elliptic second-order differential operator in divergence form has real-valued C^2 -coefficients, then the space $C_c^\infty(\mathbb{R}^d)$ is a core in $L_p(\mathbb{R}^d)$. The technique used by Wong-Dzung is then refined by Ouhabaz in [Ouh05, Theorem 5.2] to prove that $C_c^\infty(\mathbb{R}^d)$ is a core for operators in $L_2(\mathbb{R}^d)$ with real-valued $W^{2,\infty}$ -coefficients. In a recent paper [ERS11, Propositions 4.1, 4.5, 4.6 and Theorem 4.8], ter Elst, Robinson and Sikora showed the core property for the case when the coefficients are real-valued and have a mixture of smoothness between $W^{1,\infty}(\mathbb{R}^d)$ and $W^{2,\infty}(\mathbb{R}^d)$.

Apart from the interests in the core property for degenerate elliptic second-order differential operators with bounded coefficients, a large part of the literature is devoted to showing sufficient conditions under which the space of test functions is still a core for operators with real-valued coefficients which are singular either locally or at infinity. Many interesting results can be found in [Kat81], [Dav85], [Lis89], [MPPS05], [MPRS10], [CCHL12], [MS14] and references therein.

In this paper we investigate degenerate elliptic second-order differential operators with bounded complex-valued coefficients. We will provide sufficient conditions for when $C_c^\infty(\mathbb{R}^d)$ is a core for these operators. The results are generalisations of those in [WD83, Theorem I] and [Ouh05, Theorem 5.2].

Let $d \in \mathbb{N}$ and $\theta \in [0, \frac{\pi}{2})$. Let $c_{kl} \in W^{2,\infty}(\mathbb{R}^d, \mathbb{C})$ for all $k, l \in \{1, \dots, d\}$. Define $C = (c_{kl})_{1 \leq k, l \leq d}$ and $\Sigma_\theta = \{r e^{i\psi} : r \geq 0 \text{ and } |\psi| \leq \theta\}$. Assume that

$$(C(x)\xi, \xi) \in \Sigma_\theta \quad (1)$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. Later on we will usually refer to (1) as *C takes values in the sector Σ_θ* .

Define the sesquilinear form

$$\mathfrak{a}_0(u, v) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} (\partial_k u) \partial_l \bar{v}$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^d)$. Then it follows from (1) that

$$\mathfrak{a}_0(u, u) = \int_{\mathbb{R}^d} (C \nabla u, \nabla u) \in \Sigma_\theta$$

for all $u \in C_c^\infty(\mathbb{R}^d)$. Using [Kat80, Theorem VI.1.27] we deduce that \mathfrak{a}_0 is closable.

Let A be the operator associated with the closure of the form \mathfrak{a}_0 . Then $W^{2,2}(\mathbb{R}^d) \subset D(A)$ and

$$Au = - \sum_{k,l=1}^d \partial_l (c_{kl} \partial_k u)$$

for all $u \in W^{2,2}(\mathbb{R}^d)$. Furthermore, by [Kat80, Theorem VI.2.1], the operator A is an m -sectorial operator. Let S be the C_0 -semigroup generated by $-A$. If A is strongly elliptic, that is, if there exists a $\mu > 0$ such that

$$\operatorname{Re}(C(x)\xi, \xi) \geq \mu \|\xi\|^2$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, then S extends consistently to a C_0 -semigroup on $L_p(\mathbb{R}^d)$ for all $p \in [1, \infty)$ by [Aus96, Theorem 4.8]. In the general case where the coefficient matrix merely satisfies

$$(C(x)\xi, \xi) \in \Sigma_\theta$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, then we prove in Section 3 that an extension is possible for certain $p \in (1, \infty)$. Before presenting the precise statement, we need to introduce the following notation. We write

$$C = R + iB,$$

where R and B are $d \times d$ matrix-valued functions with real-valued entries. Let B_a be the anti-symmetric part of B , that is, $B_a = \frac{1}{2}(B - B^T)$. The result about semigroup extension is as follows.

Proposition 1.1. *Let $p \in (1, \infty)$. Suppose $|1 - \frac{2}{p}| \leq \cos \theta$ and $B_a = 0$. Then S extends consistently to a contraction C_0 -semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$.*

Let $p \in (1, \infty)$ be such that $|1 - \frac{2}{p}| \leq \cos \theta$. Using Proposition 1.1 we can now extend S consistently to a C_0 -semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$. Let $-A_p$ be the generator of $S^{(p)}$. Clearly $C_c^\infty(\mathbb{R}^d) \subset D(A_p)$. We wish to show that $C_c^\infty(\mathbb{R}^d)$ is a core for A_p under certain conditions on the coefficients. The first main result of this paper is as follows.

Theorem 1.2. *Let $p \in (1, \infty)$ be such that $|1 - \frac{2}{p}| < \cos \theta$. Suppose $B_a = 0$. Then the space $C_c^\infty(\mathbb{R}^d)$ is a core for A_p .*

Since A is naturally defined in $L_2(\mathbb{R}^d)$ via the closure of the form \mathfrak{a}_0 , the condition $B_a = 0$ is not needed to obtain a C_0 -semigroup on $L_2(\mathbb{R}^d)$. In this case we prove that if functions in $D(A)$ are known to possess certain smoothness properties, the space $C_c^\infty(\mathbb{R}^d)$ is always a core for A regardless of B_a .

Theorem 1.3. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then $C_c^\infty(\mathbb{R}^d)$ is a core for A .*

An overview of the contents of the subsequent sections is as follows. In Section 2 we examine the matrix of coefficients C closely. Specifically we will prove various results concerning the anti-symmetric matrix B_a . In Section 3 we prove the extension of the semigroup S to L_p -spaces. We will analyse the operator A_p in detail and then prove that $C_c^\infty(\mathbb{R}^d)$ is a core for A_p in Sections 4 and 5. In Section 6 we deal specifically with the operator A in $L_2(\mathbb{R}^d)$ and present the proof of Theorem 1.3. In Section 7 we provide some interesting examples.

2 The coefficient matrix C

Define

$$\operatorname{Re} C = \frac{C + C^*}{2} \quad \text{and} \quad \operatorname{Im} C = \frac{C - C^*}{2i},$$

where C^* is the conjugate transpose of C . Then $(\operatorname{Re} C)(x)$ and $(\operatorname{Im} C)(x)$ are self-adjoint for all $x \in \mathbb{R}^d$ and

$$C = \operatorname{Re} C + i \operatorname{Im} C. \quad (2)$$

We will also decompose the coefficient matrix C into

$$C = R + i B, \quad (3)$$

where R and B are real matrices. Write $R = R_s + R_a$, where $R_s = \frac{R+R^T}{2}$ is the symmetric part of R and $R_a = \frac{R-R^T}{2}$ is the anti-symmetric part of R . Similarly $B = B_s + B_a$, where $B_s = \frac{B+B^T}{2}$ and $B_a = \frac{B-B^T}{2}$. A comparison between (2) and (3) gives

$$\operatorname{Re} C = R_s + i B_a \quad \text{and} \quad \operatorname{Im} C = B_s - i R_a.$$

In this section we will list various relations among R_s , R_a , B_s and B_a which will be used in subsequent sections.

Lemma 2.1. *We have*

$$|(B_s \xi, \eta)| \leq \frac{1}{2} \tan \theta \left((R_s \xi, \xi) + (R_s \eta, \eta) \right)$$

for all $\xi, \eta \in \mathbb{R}^d$.

Proof. Since C takes values in Σ_θ , we have

$$|((\operatorname{Im} C(x)) \xi, \xi)| \leq \tan \theta ((\operatorname{Re} C(x)) \xi, \xi)$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. It follows that

$$|(B_s \xi, \xi)| \leq \tan \theta (R_s \xi, \xi)$$

for all $\xi \in \mathbb{R}^d$. We next use polarisation to obtain

$$|(B_s \xi, \eta)| \leq \tan \theta (R_s \xi, \xi)^{1/2} (R_s \eta, \eta)^{1/2} \leq \frac{1}{2} \tan \theta \left((R_s \xi, \xi) + (R_s \eta, \eta) \right)$$

for all $\xi, \eta \in \mathbb{R}^d$ as required. \square

Lemma 2.2. *Let $j \in \{1, \dots, d\}$. Let $f \in W^{2,\infty}(\mathbb{R}^d)$ be such that $f(x) \geq 0$ for all $x \in \mathbb{R}^d$. Then*

$$|\partial_j f|^2 \leq 2 \|\partial_j^2 f\|_\infty f.$$

Proof. Let $j \in \{1, \dots, d\}$, $x \in \mathbb{R}^d$ and $h \in \mathbb{R}$. For each $n \in \mathbb{N}$ let $f_n = J_n * f$, where J_n denotes the usual mollifier with respect to a suitable function in $C_c^\infty(\mathbb{R}^d)$. Then $f_n \geq 0$ and $f_n \in C^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Using the Taylor expansion we have

$$0 \leq f_n(x) + h (\partial_j f_n)(x) + \frac{h^2}{2} \|\partial_j^2 f_n\|_\infty$$

for all $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ we obtain

$$0 \leq f(x) + h (\partial_j f)(x) + \frac{h^2}{2} \|\partial_j^2 f\|_\infty.$$

This is true for all $h \in \mathbb{R}$. Hence $|\partial_j f(x)|^2 \leq 2 \|\partial_j^2 f\|_\infty f(x)$ as required. \square

Lemma 2.3. *Let $j \in \{1, \dots, d\}$. Let $f \in W^{2,\infty}(\mathbb{R}^d)$ be such that $f(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$. Then*

$$|\partial_j f|^2 \leq 4(1 + \tan \theta)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty \operatorname{Re} f.$$

Proof. Since $f(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$, we have $\operatorname{Re} f \geq 0$. Therefore by Lemma 2.2 we have

$$|\partial_j(\operatorname{Re} f)|^2 \leq 2 \|\partial_j^2(\operatorname{Re} f)\|_\infty \operatorname{Re} f \leq 2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty \operatorname{Re} f.$$

Also $|\operatorname{Im} f| \leq (\tan \theta) \operatorname{Re} f$. That is, $(\tan \theta) \operatorname{Re} f \pm \operatorname{Im} f \geq 0$. Applying Lemma 2.2 again we obtain

$$\begin{aligned} |\partial_j((\tan \theta) \operatorname{Re} f + \operatorname{Im} f)|^2 &\leq 2 \|\partial_j^2((\tan \theta) \operatorname{Re} f + \operatorname{Im} f)\|_\infty ((\tan \theta) \operatorname{Re} f + \operatorname{Im} f) \\ &\leq 2(1 + \tan \theta) \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty ((\tan \theta) \operatorname{Re} f + \operatorname{Im} f) \end{aligned}$$

and

$$\begin{aligned} |\partial_j((\tan \theta) \operatorname{Re} f - \operatorname{Im} f)|^2 &\leq 2 \|\partial_j^2((\tan \theta) \operatorname{Re} f - \operatorname{Im} f)\|_\infty ((\tan \theta) \operatorname{Re} f - \operatorname{Im} f) \\ &\leq 2(1 + \tan \theta) \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty ((\tan \theta) \operatorname{Re} f - \operatorname{Im} f). \end{aligned}$$

Adding the two inequalities gives

$$(\tan \theta)^2 |\partial_j(\operatorname{Re} f)|^2 + |\partial_j(\operatorname{Im} f)|^2 \leq 2(1 + \tan \theta)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty \operatorname{Re} f.$$

Hence

$$|\partial_j f|^2 = |\partial_j(\operatorname{Re} f)|^2 + |\partial_j(\operatorname{Im} f)|^2 \leq 4(1 + \tan \theta)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty \operatorname{Re} f$$

as required. □

Lemma 2.4. *Let $j \in \{1, \dots, d\}$. Let $\xi, \eta \in \mathbb{C}^d$. Then the following are valid.*

(a) $|((\partial_j C) \xi, \eta)|^2 \leq M \left(((\operatorname{Re} C) \xi, \xi) + ((\operatorname{Re} C) \eta, \eta) \right)$, where

$$M = 8(1 + \tan \theta)^2 (\|\xi\|^2 + \|\eta\|^2) \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

(b) $|((\partial_j \operatorname{Im} C) \xi, \eta)|^2 \leq M \left(((\operatorname{Re} C) \xi, \xi) + ((\operatorname{Re} C) \eta, \eta) \right)$, where

$$M = 8(1 + \tan \theta)^2 (\|\xi\|^2 + \|\eta\|^2) \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

Proof. We will prove Statement (a). The proof for Statement (b) is similar.

Since C takes values in Σ_θ , we have

$$|(C \xi, \xi)| \leq (1 + \tan \theta) ((\operatorname{Re} C) \xi, \xi).$$

Polarisation gives

$$\begin{aligned} |(C \xi, \eta)| &\leq 2(1 + \tan \theta) ((\operatorname{Re} C) \xi, \xi)^{1/2} ((\operatorname{Re} C) \eta, \eta)^{1/2} \\ &\leq (1 + \tan \theta) \left(((\operatorname{Re} C) \xi, \xi) + ((\operatorname{Re} C) \eta, \eta) \right). \end{aligned}$$

Let

$$X = (1 + \tan \theta) \left(((\operatorname{Re} C) \xi, \xi) + ((\operatorname{Re} C) \eta, \eta) \right)$$

and

$$Y = (C \xi, \eta) = Y_1 + i Y_2,$$

where Y_1 and Y_2 are real-valued functions. Since $X - Y_1 \geq 0$, it follows from Lemma 2.2 that

$$|\partial_j(X - Y_1)|^2 \leq 2 \|\partial_j^2(X - Y_1)\|_\infty (X - Y_1) \leq 2 (\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty) (X - Y_1).$$

Arguing similarly for $X + Y_1 \geq 0$ we yield

$$|\partial_j(X + Y_1)|^2 \leq 2 (\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty) (X + Y_1).$$

By adding the two inequalities we obtain

$$|\partial_j X|^2 + |\partial_j Y_1|^2 \leq 2 (\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty) X.$$

Analogously

$$|\partial_j X|^2 + |\partial_j Y_2|^2 \leq 2 (\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty) X.$$

Hence

$$\begin{aligned} |((\partial_j C) \xi, \eta)|^2 &= |\partial_j Y_1|^2 + |\partial_j Y_2|^2 \leq 4 (\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty) X \\ &\leq M \left(((\operatorname{Re} C) \xi, \xi) + ((\operatorname{Re} C) \eta, \eta) \right), \end{aligned}$$

where

$$M = 8 (1 + \tan \theta)^2 (\|\xi\|^2 + \|\eta\|^2) \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

The proof is complete. \square

Next we provide a complex version of Oleinik's inequality (cf. [Ole66]).

Proposition 2.5. *Let $j \in \{1, \dots, d\}$. Let U be a complex $d \times d$ matrix. Then the following are valid.*

$$(a) \quad |\operatorname{tr}((\partial_j C) U)|^2 \leq M \left(\operatorname{tr}(U^* (\operatorname{Re} C) U) + \operatorname{tr}(U (\operatorname{Re} C) U^*) \right), \text{ where}$$

$$M = 16 d (1 + \tan \theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

$$(b) \quad |\operatorname{tr}((\partial_j \operatorname{Im} C) U)|^2 \leq M \left(\operatorname{tr}(U^* (\operatorname{Re} C) U) + \operatorname{tr}(U (\operatorname{Re} C) U^*) \right), \text{ where}$$

$$M = 16 d (1 + \tan \theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

Proof. We will prove Statement (a). The proof for Statement (b) is similar.

Let $j \in \{1, \dots, d\}$ and

$$M = 16 d (1 + \tan \theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

Let V be a unitary matrix such that $U = V |U|$, where $|U| = \sqrt{U^* U}$. Since $|U|$ is positive and Hermitian, there exists a unitary matrix W such that $|U| = W D W^*$, where D is a positive diagonal matrix. It follows that

$$\begin{aligned}
|\operatorname{tr}((\partial_j C) U)|^2 &= |\operatorname{tr}((\partial_j C) V |U|)|^2 = |\operatorname{tr}(W^* (\partial_j C) V W W^* |U| W)|^2 \\
&= |\operatorname{tr}(W^* (\partial_j C) V W D)|^2 = \left| \sum_{k=1}^d (W^* (\partial_j C) V W)_{kk} D_{kk} \right|^2 \\
&\leq d \sum_{k=1}^d |(W^* (\partial_j C) V W)_{kk}|^2 |D_{kk}|^2 \\
&\leq M \sum_{k=1}^d \left((W^* (\operatorname{Re} C) W)_{kk} + (W^* V^* (\operatorname{Re} C) V W)_{kk} \right) |D_{kk}|^2 \\
&\leq M \sum_{k=1}^d \left(D_{kk} (W^* (\operatorname{Re} C) W)_{kk} D_{kk} + D_{kk} (W^* V^* (\operatorname{Re} C) V W)_{kk} D_{kk} \right) \\
&\leq M \left(\operatorname{tr}(|U| (\operatorname{Re} C) |U|) + \operatorname{tr}(|U| V^* (\operatorname{Re} C) V |U|) \right) \\
&= M \left(\operatorname{tr}(U (\operatorname{Re} C) U^*) + \operatorname{tr}(U^* (\operatorname{Re} C) U) \right),
\end{aligned}$$

where we used Lemma 2.4 in the second inequality. \square

Corollary 2.6. *Let $j \in \{1, \dots, d\}$. Suppose U is a complex $d \times d$ matrix with $U^T = U$. Then the following are valid.*

(a) $|\operatorname{tr}((\partial_j C) U)|^2 \leq M \operatorname{tr}(U R_s \overline{U})$, where

$$M = 32 d (1 + \tan \theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

(b) $|\operatorname{tr}((\partial_j \operatorname{Im} C) U)|^2 \leq M \operatorname{tr}(U R_s \overline{U})$, where

$$M = 32 d (1 + \tan \theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty.$$

Proof. Since $U^T = U$ we have

$$\begin{aligned}
\operatorname{tr}(U^* (\operatorname{Re} C) U) + \operatorname{tr}(U (\operatorname{Re} C) U^*) &= \operatorname{tr}(\overline{U} (\operatorname{Re} C) U) + \operatorname{tr}(U (\operatorname{Re} C) \overline{U}) \\
&= \operatorname{tr}(\overline{U} (\operatorname{Re} C) U) + \operatorname{tr}(\overline{U} (\operatorname{Re} C)^T U) \\
&= 2 \operatorname{tr}(U R_s \overline{U}).
\end{aligned}$$

Next we use Proposition 2.5 to derive the result. \square

Lemma 2.7. *Let U be a complex $d \times d$ matrix. Then*

$$((\operatorname{Re} C) U \xi, U \xi) \leq \operatorname{tr}(U^* (\operatorname{Re} C) U) \|\xi\|^2$$

for all $\xi \in \mathbb{C}^d$.

Proof. By hypothesis $\operatorname{Re} C \geq 0$. Therefore $((\operatorname{Re} C) U \xi, U \xi) \geq 0$ for all $\xi \in \mathbb{C}^d$. It follows that $U^* (\operatorname{Re} C) U \geq 0$. Hence $U^* (\operatorname{Re} C) U \leq \operatorname{tr} (U^* (\operatorname{Re} C) U) I$, where I denotes the identity matrix. This justifies the claim. \square

Lemma 2.8. *We have*

$$|(B_a \xi, \xi)| \leq (R_s \xi, \xi)$$

for all $\xi \in \mathbb{C}^d$.

Proof. Write $\xi = \xi_1 + i \xi_2$, where $\xi_1, \xi_2 \in \mathbb{R}^d$. Then $(R_s \xi, \xi) = (R_s \xi_1, \xi_1) + (R_s \xi_2, \xi_2)$ and $(B_a \xi, \xi) = -2i (B_a \xi_1, \xi_2)$. Since C takes values in Σ_θ , we have $((\operatorname{Re} C) \xi, \xi) \geq 0$ for all $\xi \in \mathbb{C}^d$. Equivalently

$$-2 (B_a \xi_1, \xi_2) \leq (R_s \xi_1, \xi_1) + (R_s \xi_2, \xi_2).$$

Replacing ξ by $\bar{\xi}$ and repeating the same process as above we also obtain

$$2 (B_a \xi_1, \xi_2) \leq (R_s \xi_1, \xi_1) + (R_s \xi_2, \xi_2).$$

The result now follows. \square

Lemma 2.9. *Let $l \in \{1, \dots, d\}$ and $\xi \in \mathbb{C}^d$. Then*

$$|((\partial_l B_a) \xi, \xi)|^2 \leq M (R_s \xi, \xi),$$

where $M = 2 \|\xi\|^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty$.

Proof. Let $l \in \{1, \dots, d\}$ and $\xi \in \mathbb{C}^d$. By Lemma 2.8 we deduce that $R_s \pm i B_a \geq 0$. Now we use Lemma 2.2 to derive

$$\begin{aligned} |(\partial_l (R_s + i B_a) \xi, \xi)|^2 &\leq 2 \|(\partial_l^2 (R_s + i B_a) \xi, \xi)\|_\infty ((R_s + i B_a) \xi, \xi) \\ &\leq 2 \|\xi\|^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty ((R_s + i B_a) \xi, \xi) \end{aligned}$$

and

$$\begin{aligned} |(\partial_l (R_s - i B_a) \xi, \xi)|^2 &\leq 2 \|(\partial_l^2 (R_s - i B_a) \xi, \xi)\|_\infty ((R_s - i B_a) \xi, \xi) \\ &\leq 2 \|\xi\|^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty ((R_s - i B_a) \xi, \xi). \end{aligned}$$

Adding the two inequalities together gives

$$|((\partial_l R_s) \xi, \xi)|^2 + |((\partial_l B_a) \xi, \xi)|^2 \leq 2 \|\xi\|^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty (R_s \xi, \xi),$$

which clearly implies the result. \square

Lemma 2.10. *Let Q be a complex $d \times d$ matrix. Suppose there exists an $M > 0$ such that $|(Q \xi, \xi)| \leq M (R_s \xi, \xi)$ for all $\xi \in \mathbb{C}^d$. Then $\|Q \xi\|^2 \leq 4 M^2 \|R_s\|_\infty (R_s \xi, \xi)$ for all $\xi \in \mathbb{C}^d$.*

Proof. Since $|(Q \xi, \xi)| \leq M (R_s \xi, \xi)$ for all $\xi \in \mathbb{C}^d$, polarisation gives

$$|(Q \xi, \eta)| \leq 2 M (R_s \xi, \xi)^{1/2} (R_s \eta, \eta)^{1/2} \leq 2 M \|R_s\|_\infty^{1/2} \|\eta\| (R_s \xi, \xi)^{1/2}$$

for all $\xi, \eta \in \mathbb{C}^d$. It follows that

$$\|Q \xi\| \leq 2 M \|R_s\|_\infty^{1/2} (R_s \xi, \xi)^{1/2}$$

for all $\xi \in \mathbb{C}^d$, which justifies the claim. \square

Lemma 2.11. *We have*

$$\|C\xi\|^2 \leq 16(1 + \tan\theta)^2 \|R_s\|_\infty (R_s\xi, \xi)$$

for all $\xi \in \mathbb{C}^d$.

Proof. Let $\xi \in \mathbb{C}^d$. Since C takes values in Σ_θ , we have

$$|(C\xi, \xi)| \leq ((\operatorname{Re} C)\xi, \xi) + |((\operatorname{Im} C)\xi, \xi)| \leq (1 + \tan\theta) ((\operatorname{Re} C)\xi, \xi).$$

However $((\operatorname{Re} C)\xi, \xi) \leq 2(R_s\xi, \xi)$ by Lemma 2.8. It follows that

$$|(C\xi, \xi)| \leq 2(1 + \tan\theta) (R_s\xi, \xi).$$

Using Lemma 2.10 we obtain

$$\|C\xi\|^2 \leq 16(1 + \tan\theta)^2 \|R_s\|_\infty (R_s\xi, \xi)$$

as required. \square

Recall that the Hilbert-Schmidt norm for a matrix $V \in M_{d \times d}(\mathbb{C})$ is defined by

$$\|V\|_{HS} = (\operatorname{tr}(V^*V))^{1/2} = \left(\sum_{j=1}^d \|Ve_j\|^2 \right)^{1/2}.$$

Lemma 2.12. *Let U a complex $d \times d$ matrix with $U^T = U$. Then*

$$\|CU\|_{HS}^2 \leq 16(1 + \tan\theta)^2 \|R_s\|_\infty \operatorname{tr}(UR_s\overline{U}).$$

Proof. We note that

$$\begin{aligned} \|CU\|_{HS}^2 &= \sum_{j=1}^d \|CUe_j\|_2^2 \leq 16(1 + \tan\theta)^2 \|R_s\|_\infty \sum_{j=1}^d (R_s Ue_j, Ue_j) \\ &= 16(1 + \tan\theta)^2 \|R_s\|_\infty \operatorname{tr}(UR_s\overline{U}), \end{aligned}$$

where we used Lemma 2.11 in the second step. \square

3 L_p extension

Let S be the contraction C_0 -semigroup generated by $-A$. In this section we will extend S to a contraction C_0 -semigroup on $L_p(\mathbb{R}^d)$ for all $p \in (1, \infty)$ with $|1 - \frac{2}{p}| \leq \cos\theta$, under the condition that $B_a = 0$.

Proof of Proposition 1.1. We proceed via two steps.

Step 1: Suppose that A is strongly elliptic.

Then S extends consistently to a C_0 -semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$ by [AMT98, Theorem 2.21]. Using duality arguments we can assume without loss of generality that $p \geq 2$. Let $-A_p$ be the generator of $S^{(p)}$. Let $u \in \mathcal{D}$, where $\mathcal{D} = D(A) \cap D(A_p) \cap L_\infty(\mathbb{R}^d)$. Since A is strongly elliptic, the form \mathfrak{a}_0 is closable and $D(\overline{\mathfrak{a}}_0) = W^{1,2}(\mathbb{R}^d)$. By construction $D(A) \subset D(\overline{\mathfrak{a}}_0)$.

Therefore $u \in W^{1,2}(\mathbb{R}^d)$. Set $v = |u|^{p-2}u$. Then $v \in L_q(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, where q is the dual exponent of p . By [GT83, Lemma 7.7] we have

$$\partial_l v = \frac{p}{2} |u|^{p-2} \partial_l u + \frac{p-2}{2} |u|^{p-4} u^2 \partial_l \bar{u}$$

for all $l \in \{1, \dots, d\}$. It follows that $v \in W^{1,2}(\mathbb{R}^d)$. Our aim is to prove the inequality $\operatorname{Re} \int (A_p u) \bar{v} \geq 0$, where here and in the rest of this paragraph the integral is over the set $\{x \in \mathbb{R}^d : u(x) \neq 0\}$. Indeed we have

$$\begin{aligned} \int (A_p u) \bar{v} &= \int (A u) \bar{v} = \overline{\mathfrak{a}_0}(u, v) = \sum_{k,l=1}^d \int c_{kl} (\partial_k u) \partial_l \bar{v} \\ &= \sum_{k,l=1}^d \int c_{kl} (\partial_k u) \left(\frac{p}{2} |u|^{p-2} \partial_l \bar{u} + \frac{p-2}{2} |u|^{p-4} \bar{u}^2 \partial_l u \right) \\ &= \frac{1}{2} \int |u|^{p-4} \sum_{k,l=1}^d \left(p c_{kl} |u|^2 (\partial_k u) \partial_l \bar{u} + (p-2) c_{kl} \bar{u}^2 (\partial_k u) \partial_l u \right) \\ &= \frac{1}{2} \int |u|^{p-4} \left(p (C u \nabla \bar{u}, u \nabla \bar{u}) + (p-2) (C \bar{u} \nabla u, u \nabla \bar{u}) \right). \end{aligned}$$

Write $u \nabla \bar{u} = \xi + i \eta$, where $\xi, \eta \in \mathbb{R}^d$. Then

$$\operatorname{Re} (C u \nabla \bar{u}, u \nabla \bar{u}) = (R_s \xi, \xi) + (R_s \eta, \eta) + 2 (B_a \xi, \eta) = (R_s \xi, \xi) + (R_s \eta, \eta)$$

as $B_a = 0$ by hypothesis and

$$\operatorname{Re} (C \bar{u} \nabla u, u \nabla \bar{u}) = (R_s \xi, \xi) - (R_s \eta, \eta) + 2 (B_s \xi, \eta).$$

Therefore

$$\begin{aligned} \operatorname{Re} \int (A_p u) \bar{v} &= \int |u|^{p-4} \left((p-1) (R_s \xi, \xi) + (R_s \eta, \eta) + (p-2) (B_s \xi, \eta) \right) \\ &= \int |u|^{p-4} \left((R_s \xi', \xi') + (R_s \eta, \eta) + \frac{p-2}{\sqrt{p-1}} (B_s \xi', \eta) \right), \end{aligned}$$

where $\xi' = \sqrt{p-1} \xi$. If $\theta = 0$ then it follows from Lemma 2.1 that $(B_s \xi', \eta) = 0$. Consequently

$$\operatorname{Re} \int (A_p u) \bar{v} = \int |u|^{p-4} \left((R_s \xi', \xi') + (R_s \eta, \eta) \right) \geq 0.$$

If $\theta \neq 0$ then

$$\operatorname{Re} \int (A_p u) \bar{v} \geq \int |u|^{p-4} \left((R_s \xi', \xi') + (R_s \eta, \eta) - 2 \cot \theta |(B_s \xi', \eta)| \right) \geq 0,$$

where we again used Lemma 2.1 and the fact that $|1 - \frac{2}{p}| \leq \cos \theta$ is equivalent to $|p-2| \tan \theta \leq 2\sqrt{p-1}$. In either case the restriction $A_p|_{\mathcal{D}}$ is accretive. Since \mathcal{D} is a core for A_p , we also have that A_p is accretive by [LP61, Lemma 3.4]. By the Lumer-Phillips theorem, $S^{(p)}$ is a contraction semigroup.

Step 2: Suppose that A is degenerate elliptic.

Let $n \in \mathbb{N}$. Let $A_{[n]} = A - \frac{1}{n} \Delta$, where $\Delta = \partial_1^2 + \dots + \partial_d^2$. Then $A_{[n]}$ is strongly elliptic. Let $S^{[n]}$ be the contraction C_0 -semigroup generated by $A_{[n]}$. Then $S^{[n]}$ extends consistently to a contraction C_0 -semigroup $S^{(n,p)}$ on $L_p(\mathbb{R}^d)$ by Step 1. Using duality arguments we can assume without loss of generality that $p \in (1, 2)$.

Let $t > 0$ and $u \in L_{2,c}(\mathbb{R}^d)$. By [AE12, Corollary 3.9] we have $\lim_{n \rightarrow \infty} S_t^{[n]} u = S_t u$ in $L_2(\mathbb{R}^d)$. Also by [AE12, Lemma 4.5] we obtain $\lim_{n \rightarrow \infty} S_t^{[n]} u = S_t u$ in $L_1(\mathbb{R}^d)$. Interpolation then gives $\lim_{n \rightarrow \infty} S_t^{[n]} u = S_t u$ in $L_p(\mathbb{R}^d)$. It follows that $\|S_t u\|_p \leq \|u\|_p$ as $S^{(n,p)}$ is contractive on $L_p(\mathbb{R}^d)$. But $L_{2,c}(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$. Therefore $\|S_t u\|_p \leq \|u\|_p$ for all $u \in L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$. That is, $S_t|_{L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)}$ extends continuously to a contraction operator $S_t^{(p)}$ on $L_p(\mathbb{R}^d)$. We now use [Voi92, Proposition 1] to conclude that $S^{(p)}$ is a C_0 -semigroup on $L_p(\mathbb{R}^d)$. \square

4 The operator B_p

Let $p \in (1, \infty)$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Define

$$H_q u = - \sum_{k,l=1}^d \partial_k (\overline{c_{kl}} \partial_l u) \quad (4)$$

on the domain

$$D(H_q) = C_c^\infty(\mathbb{R}^d).$$

Next define $B_p = (H_q)^*$, which is the dual of H_q . Then B_p is closed by [Kat80, Subsection III.5.5]. Also note that $W^{2,p}(\mathbb{R}^d) \subset D(B_p)$ and

$$B_p u = - \sum_{k,l=1}^d \partial_l (c_{kl} \partial_k u)$$

for all $u \in W^{2,p}(\mathbb{R}^d)$.

We will prove at the end of this section that $C_c^\infty(\mathbb{R}^d)$ is a core for B_p if $|1 - \frac{2}{p}| < \cos \theta$ and $B_a = 0$. In the next section we will prove that $A_p = B_p$ under the same assumptions. The proofs require a lot of preparation.

Proposition 4.1. *Suppose $|1 - \frac{2}{p}| \leq \cos \theta$ and $B_a = 0$. Then*

$$\operatorname{Re} (B_p u, |u|^{p-2} u) \geq 0$$

for all $u \in W^{2,p}(\mathbb{R}^d)$.

Proof. Let $u \in W^{2,p}(\mathbb{R}^d)$. It follows from the proof of [MS08, Proposition 3.5] that

$$\begin{aligned} (B_p u, |u|^{p-2} u) &= \int_{[u \neq 0]} |u|^{p-2} (C \nabla \overline{u}, \nabla \overline{u}) \\ &\quad + (p-2) \int_{[u \neq 0]} |u|^{p-4} (C \operatorname{Re} (u \nabla \overline{u}), \operatorname{Re} (u \nabla \overline{u})) \\ &\quad - i (p-2) \int_{[u \neq 0]} |u|^{p-4} (C \operatorname{Re} (u \nabla \overline{u}), \operatorname{Im} (u \nabla \overline{u})). \end{aligned} \quad (5)$$

Write $u \nabla \bar{u} = \xi + i \eta$, where $\xi, \eta \in \mathbb{R}^d$. Then

$$\begin{aligned} |u|^2 (C \nabla \bar{u}, \nabla \bar{u}) &= (C u \nabla \bar{u}, u \nabla \bar{u}) = (C(\xi + i \eta), \xi + i \eta) \\ &= (R \xi, \xi) + (R \eta, \eta) + (B \xi, \eta) - (B \eta, \xi) \\ &\quad - i ((R \eta, \xi) - (R \xi, \eta) + (B \xi, \xi) + (B \eta, \eta)). \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{Re} (|u|^2 (C \nabla \bar{u}, \nabla \bar{u})) &= (R \xi, \xi) + (R \eta, \eta) + (B \xi, \eta) - (B \eta, \xi) \\ &= (R_s \xi, \xi) + (R_s \eta, \eta) + 2 (B_a \xi, \eta) \\ &= (R_s \xi, \xi) + (R_s \eta, \eta) \end{aligned}$$

since $B_a = 0$. We also have

$$\operatorname{Re} (C \operatorname{Re} (u \nabla \bar{u}), \operatorname{Re} (u \nabla \bar{u})) = \operatorname{Re} (C \xi, \xi) = (R \xi, \xi) = (R_s \xi, \xi).$$

Similarly

$$\operatorname{Re} (i (C \operatorname{Re} (u \nabla \bar{u}), \operatorname{Im} (u \nabla \bar{u}))) = \operatorname{Re} (i (C \xi, \eta)) = -(B \xi, \eta) = -(B_s \xi, \eta)$$

since $B_a = 0$. Hence taking the real parts on both sides of (5) yields

$$\operatorname{Re} (B_p u, |u|^{p-2} u) = \int_{[u \neq 0]} |u|^{p-4} \left((p-1) (R_s \xi, \xi) + (R_s \eta, \eta) + (p-2) (B_s \xi, \eta) \right)$$

since $B_a = 0$. Now we argue as in Step 1 of the proof of Proposition 1.1 to derive the claim. \square

Let $J \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ be such that $J \geq 0$, $\operatorname{supp} J \subset B_1(0)$ and $\int_{\mathbb{R}^d} J = 1$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ define $J_n(x) = n^d J(nx)$. For all $n \in \mathbb{N}$ define the bounded operator $T_n^{(1)} : W^{1,p}(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ by

$$T_n^{(1)} u = - \sum_{k,l=1}^d \int_{\mathbb{R}^d} J_n(y) \left((I - L_y) (\partial_l c_{kl}) \right) L_y (\partial_k u) dy$$

and the bounded operator $T_n^{(2)} : W^{1,p}(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ by

$$T_n^{(2)} u = - \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial y_l} \left(J_n(y) (I - L_y) c_{kl} \right) \right) L_y (\partial_k u) dy,$$

where $(L_y u)(x) = u(x - y)$ for all $x, y \in \mathbb{R}^d$. Also define for all $n \in \mathbb{N}$ the operator $T_n : W^{1,p}(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ by

$$T_n = T_n^{(1)} + T_n^{(2)}. \tag{6}$$

Lemma 4.2. *The sequence $\{T_n^{(1)}\}_{n \in \mathbb{N}}$ is bounded. Furthermore $\lim_{n \rightarrow \infty} \|T_n^{(1)} u\|_p = 0$ for all $u \in W^{1,p}(\mathbb{R}^d)$.*

Proof. Let $n \in \mathbb{N}$ and $u \in W^{1,p}(\mathbb{R}^d)$. For all $k, l \in \{1, \dots, d\}$ we have $c_{kl} \in W^{2,\infty}(\mathbb{R}^d)$, which implies

$$|(\partial_l c_{kl})(x) - (\partial_l c_{kl})(x - y)| \leq \|c_{kl}\|_{W^{2,\infty}} |y| \quad (7)$$

for all $x, y \in \mathbb{R}^d$. It follows that

$$\begin{aligned} \|T_n^{(1)}u\|_p &\leq \sum_{k,l=1}^d \int_{\mathbb{R}^d} |J_n(y)| \left\| \left((I - L_y)(\partial_l c_{kl}) \right) L_y(\partial_k u) \right\|_p dy \\ &\leq \sum_{k,l=1}^d \int_{\mathbb{R}^d} |J_n(y)| \|(I - L_y)(\partial_l c_{kl})\|_\infty \|L_y(\partial_k u)\|_p dy \\ &\leq \left(\sum_{k,l=1}^d \|c_{kl}\|_{W^{2,\infty}} \right) \|u\|_{W^{1,p}} \int_{\mathbb{R}^d} |J_n(y)| |y| dy \\ &= \left(\sum_{k,l=1}^d \|c_{kl}\|_{W^{2,\infty}} \right) \|u\|_{W^{1,p}} \frac{1}{n} \int_{\mathbb{R}^d} |J(y)| |y| dy, \end{aligned}$$

where we used $J_n(y) = n^d J(ny)$ in the last step. Note that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathbb{R}^d} |J(y)| |y| dy = 0.$$

Hence $\lim_{n \rightarrow \infty} \|T_n^{(1)}u\|_p = 0$. Moreover, $\{T_n^{(1)}\}_{n \in \mathbb{N}}$ is bounded. \square

Lemma 4.3. *The sequence $\{T_n^{(2)}\}_{n \in \mathbb{N}}$ is bounded. Furthermore $\lim_{n \rightarrow \infty} \|T_n^{(2)}u\|_p = 0$ for all $u \in W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$.*

Proof. Let $n \in \mathbb{N}$ and $u \in W^{1,p}(\mathbb{R}^d)$. Expanding $T_n^{(2)}$ gives

$$T_n^{(2)}u = - \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(J_n(y) L_y(\partial_l c_{kl}) + (\partial_l J_n)(y) (I - L_y) c_{kl} \right) L_y(\partial_k u) dy,$$

where we used $L_y(\partial_l c_{kl}) = -\frac{\partial}{\partial y_l}(L_y c_{kl})$ for all $k, l \in \{1, \dots, d\}$. Therefore

$$\begin{aligned} \|T_n^{(2)}u\|_p &\leq \sum_{k,l=1}^d \left(\|(\partial_l c_{kl})(\partial_k u)\|_p + \int_{\mathbb{R}^d} |(\partial_l J_n)(y)| \|((I - L_y) c_{kl}) L_y(\partial_k u)\|_p dy \right) \\ &\leq \sum_{k,l=1}^d \left(\|\partial_l c_{kl}\|_\infty \|(\partial_k u)\|_p + \int_{\mathbb{R}^d} |(\partial_l J_n)(y)| \|(I - L_y) c_{kl}\|_\infty \|L_y(\partial_k u)\|_p dy \right) \\ &\leq M \|u\|_{W^{1,p}}, \end{aligned} \quad (8)$$

where

$$M = \sum_{k,l=1}^d \left(\|c_{kl}\|_{W^{2,\infty}} \left(1 + \int_{\mathbb{R}^d} |(\partial_l J)(y)| |y| dy \right) \right) \quad (9)$$

and we used (7) in the last step. Therefore $\{T_n^{(2)}\}_{n \in \mathbb{N}}$ is bounded.

To prove to the latter statement of the lemma, we consider two cases.

Case 1: Suppose $u \in C_c^\infty(\mathbb{R}^d)$.

Since J_n has a compact support, we have

$$\sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial y_l} \left(J_n(y) (I - L_y) c_{kl} \right) \right) (\partial_k u) dy = 0.$$

Consequently

$$\begin{aligned} T_n^{(2)} u &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial y_l} \left(J_n(y) (I - L_y) c_{kl} \right) \right) (I - L_y) (\partial_k u) dy \\ &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(J_n(y) L_y (\partial_l c_{kl}) + (\partial_l J_n)(y) (I - L_y) c_{kl} \right) (I - L_y) (\partial_k u) dy. \end{aligned}$$

It follows that

$$\|T_n^{(2)} u\|_p \leq \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(J_n(y) \|L_y (\partial_l c_{kl})\|_\infty + |(\partial_l J_n)(y)| \|(I - L_y) c_{kl}\|_\infty \right) \|(I - L_y) (\partial_k u)\|_p dy.$$

Note that

$$\begin{aligned} \|(I - L_y) (\partial_k u)\|_p &= \left(\int_{\mathbb{R}^d} |(\partial_k u)(x) - (\partial_k u)(x - y)|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}^d} (\|u\|_{W^{2,\infty}} |y|)^p \mathbf{1}_{\text{supp } \partial_k u \cup \text{supp } L_y (\partial_k u)} dx \right)^{\frac{1}{p}} \\ &\leq 2 |\text{supp } \partial_k u|^{1/p} \|u\|_{W^{2,\infty}} |y| \leq \frac{2}{n} |\text{supp } u|^{1/p} \|u\|_{W^{2,\infty}} \end{aligned}$$

for all $k \in \{1, \dots, d\}$ and $y \in \mathbb{R}^d$ such that $|y| < \frac{1}{n}$, where in the last step we used the fact that $\text{supp } \partial_k u \subset \text{supp } u$ for all $k \in \{1, \dots, d\}$. Therefore

$$\|T_n^{(2)} u\|_p \leq \frac{2M}{n} |\text{supp } u|^{1/p} \|u\|_{W^{2,\infty}}, \quad (10)$$

where M is defined by (9) and we used the fact that $\int_{\mathbb{R}^d} J_n(y) dy = 1$. Hence (10) gives $\lim_{n \rightarrow \infty} \|T_n^{(2)} u\|_p = 0$.

Case 2: Suppose $u \in W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$.

Let $\varepsilon > 0$. Let $v \in C_c^\infty(\mathbb{R}^d)$ be such that $\|u - v\|_{W^{1,p}} < \frac{\varepsilon}{2M}$. Choose an $n \in \mathbb{N}$ such that $\frac{2M}{n} |\text{supp } v|^{1/p} \|v\|_{W^{2,\infty}} < \frac{\varepsilon}{2}$. Then it follows from (8) and (10) that

$$\|T_n^{(2)} u\|_p \leq \|T_n^{(2)} (u - v)\|_p + \|T_n^{(2)} v\|_p \leq M \|u - v\|_{W^{1,p}} + \frac{2M}{n} |\text{supp } v|^{1/p} \|v\|_{W^{2,\infty}} < \varepsilon.$$

The proof is complete. \square

Lemma 4.4. *The sequence $\{T_n\}_{n \in \mathbb{N}}$ is bounded. Furthermore $\lim_{n \rightarrow \infty} \|T_n u\|_p = 0$ for all $u \in W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$.*

Proof. This is a consequence of Lemmas 4.2 and 4.3. \square

We have the following approximation proposition (cf. [Fri44] and [Kat72] for a special case of the proposition when the coefficient c_{kl} are real-valued for all $k, l \in \{1, \dots, d\}$).

Proposition 4.5. *Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$. Then $\lim_{n \rightarrow \infty} B_p(J_n * u) = B_p u$ in $L_p(\mathbb{R}^d)$.*

Proof. Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$. It is well-known that $\lim_{n \rightarrow \infty} J_n * (B_p u) = B_p u$ in $L_p(\mathbb{R}^d)$. Therefore it suffices to show that

$$\lim_{n \rightarrow \infty} \|B_p(J_n * u) - J_n * (B_p u)\|_p = 0.$$

In what follows note that $L_y(\partial_l u) = -\frac{\partial}{\partial_l}(L_y u)$ and $\partial_l(J_n * u) = (\partial_l J_n) * u$ for all $l \in \{1, \dots, d\}$. We first calculate $J_n * (B_p u)$. Let $x \in \mathbb{R}^d$. Define $\phi(y) = J_n(x - y)$ for all $y \in \mathbb{R}^d$. Then $\phi \in C_c^\infty(\mathbb{R}^d)$. By the definition of B_p we have

$$\begin{aligned} (J_n * (B_p u))(x) &= \int_{\mathbb{R}^d} J_n(x - y) (B_p u)(y) dy = (B_p u, \phi) = (u, H_q \phi) \\ &= - \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial y_k} (c_{kl}(y)) \frac{\partial}{\partial y_l} J_n(x - y) \right) u(y) dy \\ &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(c_{kl}(y) \frac{\partial}{\partial y_l} J_n(x - y) \right) (\partial_k u)(y) dy \\ &= - \sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l J_n)(x - y) (c_{kl} \partial_k u)(y) dy \\ &= - \sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l J_n)(y) (c_{kl} \partial_k u)(x - y) dy \end{aligned}$$

for all $n \in \mathbb{N}$. Hence

$$J_n * (B_p u) = - \sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l J_n)(y) L_y(c_{kl} \partial_k u) dy$$

for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. We have

$$\begin{aligned} &B_p(J_n * u) - J_n * (B_p u) \\ &= - \sum_{k,l=1}^d \left(\partial_l \left(c_{kl} \int_{\mathbb{R}^d} J_n(y) L_y(\partial_k u) dy \right) - \int_{\mathbb{R}^d} (\partial_l J_n)(y) L_y(c_{kl} \partial_k u) dy \right) \\ &= - \sum_{k,l=1}^d \left((\partial_l c_{kl}) \int_{\mathbb{R}^d} J_n(y) L_y(\partial_k u) dy + c_{kl} \partial_l \left(\int_{\mathbb{R}^d} J_n(y) L_y(\partial_k u) dy \right) \right. \\ &\quad \left. - \int_{\mathbb{R}^d} (\partial_l J_n)(y) L_y(c_{kl} \partial_k u) dy \right) \end{aligned}$$

$$\begin{aligned}
&= - \sum_{k,l=1}^d \left((\partial_l c_{kl}) \int_{\mathbb{R}^d} J_n(y) L_y(\partial_k u) dy + c_{kl} \int_{\mathbb{R}^d} (\partial_l J_n)(y) L_y(\partial_k u) dy \right. \\
&\quad \left. - \int_{\mathbb{R}^d} (\partial_l J_n)(y) L_y(c_{kl} \partial_k u) dy \right).
\end{aligned}$$

On the other hand expanding $T_n^{(1)}$ and $T_n^{(2)}$ gives

$$T_n^{(1)} u = - \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(J_n(y) (\partial_l c_{kl}) L_y(\partial_k u) - J_n(y) L_y((\partial_l c_{kl}) \partial_k u) \right) dy$$

and

$$\begin{aligned}
T_n^{(2)} u &= - \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(J_n(y) L_y(\partial_l c_{kl}) + (\partial_l J_n)(y) (I - L_y) c_{kl} \right) L_y(\partial_k u) dy \\
&= - \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(J_n(y) L_y((\partial_l c_{kl}) \partial_k u) + (\partial_l J_n)(y) c_{kl} L_y(\partial_k u) \right. \\
&\quad \left. - (\partial_l J_n)(y) L_y(c_{kl} \partial_k u) \right) dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
T_n u &= T_n^{(1)} u + T_n^{(2)} u = - \sum_{k,l=1}^d \left((\partial_l c_{kl}) \int_{\mathbb{R}^d} J_n(y) L_y(\partial_k u) dy + c_{kl} \int_{\mathbb{R}^d} (\partial_l J_n)(y) L_y(\partial_k u) dy \right. \\
&\quad \left. - \int_{\mathbb{R}^d} (\partial_l J_n)(y) L_y(c_{kl} \partial_k u) dy \right).
\end{aligned}$$

Hence

$$B_p(J_n * u) - J_n * (B_p u) = T_n u. \quad (11)$$

The claim now follows from Lemma 4.4. \square

Let $\tau \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \leq \tau \leq \mathbb{1}$, $\tau|_{B_1(0)} = 1$ and $\text{supp } \tau \subset B_2(0)$. Define $\tau_n(x) = \tau(n^{-1}x)$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Lemma 4.6. *Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d)$. Then $\tau_n u \in D(B_p)$ for all $n \in \mathbb{N}$ and we have $\lim_{n \rightarrow \infty} \tau_n u = u$ in $D(B_p)$. If u satisfies further that $u \in W^{2,p}(\mathbb{R}^d)$ and $\nabla(B_p u) \in (L_p(\mathbb{R}^d))^d$, then $\nabla(B_p(\tau_n u)) \in (L_p(\mathbb{R}^d))^d$ and $\lim_{n \rightarrow \infty} \nabla(B_p(\tau_n u)) = \nabla(B_p u)$ in $(L_p(\mathbb{R}^d))^d$.*

Proof. Let $n \in \mathbb{N}$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$(\tau_n u, H_q \phi) = (v, \phi),$$

where

$$v = \tau_n(B_p u) + (B_p \tau_n) u - \sum_{k,l=1}^d c_{kl} (\partial_k u) (\partial_l \tau_n) - \sum_{k,l=1}^d c_{kl} (\partial_l u) (\partial_k \tau_n). \quad (12)$$

It follows that

$$\|v\|_p \leq M_1 \|u\|_{W^{1,p}} + \|B_p u\|_p < \infty,$$

where $M_1 = 3 \sup\{\|c_{kl} \tau\|_{W^{2,\infty}} : 1 \leq k, l \leq d\}$. Therefore $\tau_n u \in D(B_p)$ and $B_p(\tau_n u) = v$.

Next we consider the expression for v in (12). For the first term we have $\|\tau_n(B_p u)\|_p \leq \|B_p u\|$ for all $n \in \mathbb{N}$ and $\{\tau_n(B_p u)\}_{n \in \mathbb{N}}$ converges to $B_p u$ pointwise. As a consequence $\lim_{n \rightarrow \infty} \tau_n(B_p u) = B_p u$ in $L_p(\mathbb{R}^d)$ by the Lebesgue dominated convergence theorem. For the second term we notice that

$$(\partial_k \tau_n)(x) = \frac{1}{n}(\partial_k \tau)(n^{-1}x) \quad \text{and} \quad (\partial_l \partial_k \tau_n)(x) = \frac{1}{n^2}(\partial_l \partial_k \tau)(n^{-1}x) \quad (13)$$

for all $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $k, l \in \{1, \dots, d\}$. Since $c_{kl} \in W^{2,\infty}(\mathbb{R}^d)$ for all $k, l \in \{1, \dots, d\}$, we obtain

$$\|(B_p \tau_n) u\|_p = \left\| \left(\sum_{k,l=1}^d (\partial_l c_{kl}) (\partial_k \tau_n) + c_{kl} (\partial_l \partial_k \tau_n) \right) u \right\|_p \leq \frac{2d^2}{n} \|c_{kl}\|_{W^{2,\infty}} \|\tau\|_{W^{2,\infty}} \|u\|_p \quad (14)$$

for all $n \in \mathbb{N}$. It follows that $\lim_{n \rightarrow \infty} \|(B_p \tau_n) u\|_p = 0$. Similarly the last two terms also converge to 0 in $L_p(\mathbb{R}^d)$. Clearly $\lim_{n \rightarrow \infty} \tau_n u = u$ in $L_p(\mathbb{R}^d)$. Hence $\lim_{n \rightarrow \infty} \tau_n u = u$ in $D(B_p)$.

To prove the second statement let $j \in \{1, \dots, d\}$ and $n \in \mathbb{N}$. Using (12) we have

$$\begin{aligned} \partial_j(B_p(\tau_n u)) &= \tau_n \partial_j(B_p u) + (\partial_j \tau_n)(B_p u) + (B_p \tau_n)(\partial_j u) + (\partial_j(B_p \tau_n)) u \\ &\quad - \sum_{k,l=1}^d (\partial_j c_{kl}) (\partial_k u) (\partial_l \tau_n) + c_{kl} (\partial_j \partial_k u) (\partial_l \tau_n) + c_{kl} (\partial_k u) (\partial_j \partial_l \tau_n) \\ &\quad - \sum_{k,l=1}^d (\partial_j c_{kl}) (\partial_l u) (\partial_k \tau_n) + c_{kl} (\partial_j \partial_l u) (\partial_k \tau_n) + c_{kl} (\partial_l u) (\partial_j \partial_k \tau_n). \end{aligned} \quad (15)$$

It follows that

$$\|\partial_j(B_p(\tau_n u))\|_p \leq M_2 \|u\|_{W^{2,p}} + (1 \wedge \|\tau\|_{W^{1,\infty}}) \|B_p u\|_{W^{1,p}} < \infty,$$

where $M_2 = 8 \sup\{\|c_{kl}\|_{W^{2,\infty}} \|\tau\|_{W^{3,\infty}} : 1 \leq k, l \leq d\}$. Therefore $\partial_j(B_p(\tau_n u)) \in L_p(\mathbb{R}^d)$. Furthermore notice that

$$(\partial_j \partial_l \partial_k \tau_n)(x) = \frac{1}{n^3}(\partial_j \partial_l \partial_k \tau)(n^{-1}x) \quad (16)$$

for all $x \in \mathbb{R}^d$ and $k, l \in \{1, \dots, d\}$. Using (13), (16) and repeating the arguments used in (14) we see that all terms in the expression for $\partial_j(B_p(\tau_n u))$ in (15) converge to 0 in $L_p(\mathbb{R}^d)$ as n tends to infinity except for the first one, whereas the first term converges to $\partial_j(B_p u)$ in $L_p(\mathbb{R}^d)$ as n tends to infinity. Hence $\lim_{n \rightarrow \infty} \partial_j(B_p(\tau_n u)) = \partial_j(B_p u)$ in $L_p(\mathbb{R}^d)$. This completes the proof. \square

Proposition 4.7. *The space $C_c^\infty(\mathbb{R}^d)$ is dense in $(D(B_p) \cap W^{1,p}(\mathbb{R}^d), \|\cdot\|_{D(B_p)})$.*

Proof. Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d)$ and $\varepsilon > 0$. For all $n \in \mathbb{N}$ set $u_n = \tau_n u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$. By Lemma 4.6 we can choose an $n \in \mathbb{N}$ such that $\|u - u_n\|_{D(B_p)} < \frac{\varepsilon}{2}$. Next for all $m \in \mathbb{N}$ set $v_m = J_m * (\tau_n u) \in C_c^\infty(\mathbb{R}^d)$. We now use Lemma 4.5 to choose an $m \in \mathbb{N}$ such that $\|u_n - v_m\|_{D(B_p)} < \frac{\varepsilon}{2}$. Then

$$\|u - v_m\|_{D(B_p)} \leq \|u - u_n\|_{D(B_p)} + \|u_n - v_m\|_{D(B_p)} < \varepsilon.$$

This verifies the claim. \square

Proposition 4.8. Suppose $|1 - \frac{2}{p}| < \cos \theta$ and $B_a = 0$. Then there exists an $M > 0$ such that

$$\operatorname{Re}(\nabla(B_p u), |\nabla u|^{p-2} \nabla u) \geq -M \|\nabla u\|_p^p$$

for all $u \in W^{2,p}(\mathbb{R}^d)$ such that $\nabla(B_p u) \in (L_p(\mathbb{R}^d))^d$.

Proof. The condition $|1 - \frac{2}{p}| < \cos \theta$ is equivalent to $|p - 2| \tan \theta < 2\sqrt{p-1}$. Let $\varepsilon_0 \in (0, 1 \wedge (p-1))$ be such that

$$|p - 2| \tan \theta \leq 2\sqrt{(1-\varepsilon)(p-1-\varepsilon)}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Let $\varepsilon \in (0, \varepsilon_0)$ be such that

$$\varepsilon < \frac{\varepsilon_0}{32 d (1 + \tan \theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty}. \quad (17)$$

Let $u \in W^{2,p}(\mathbb{R}^d)$. By Lemma 4.6 we can assume without loss of generality that u has a compact support. For the rest of the proof, all integrations are over the set $\{x \in \mathbb{R}^d : |(\nabla u)(x)| \neq 0\}$. We have

$$\begin{aligned} (\nabla(B_p u), |\nabla u|^{p-2} \nabla u) &= - \sum_{k,l,j=1}^d \int \left(\partial_j \partial_l (c_{kl} \partial_k u) \right) |\nabla u|^{p-2} \partial_j \bar{u} \\ &= - \sum_{k,l,j=1}^d \int \left(\partial_l ((\partial_j c_{kl}) (\partial_k u) + c_{kl} (\partial_j \partial_k u)) \right) |\nabla u|^{p-2} \partial_j \bar{u} \\ &= - \sum_{k,l,j=1}^d \int \left(\partial_l ((\partial_j c_{kl}) (\partial_k u)) \right) |\nabla u|^{p-2} \partial_j \bar{u} \\ &\quad + \sum_{k,l,j=1}^d \int c_{kl} (\partial_j \partial_k u) \partial_l (|\nabla u|^{p-2} \partial_j \bar{u}) \\ &= \text{(I)} + \text{(II)}. \end{aligned}$$

We first consider the real part of (I). We have

$$\begin{aligned} -\operatorname{Re} \sum_{k,l,j=1}^d \int \left(\partial_l ((\partial_j c_{kl}) (\partial_k u)) \right) |\nabla u|^{p-2} \partial_j \bar{u} &= -\operatorname{Re} \sum_{k,l,j=1}^d \int (\partial_l \partial_j c_{kl}) (\partial_k u) (\partial_j \bar{u}) |\nabla u|^{p-2} \\ &\quad - \operatorname{Re} \sum_{k,l,j=1}^d \int (\partial_j c_{kl}) (\partial_l \partial_k u) (\partial_j \bar{u}) |\nabla u|^{p-2} \\ &= \text{(Ia)} + \text{(Ib)}. \end{aligned}$$

For (Ia) we have

$$\text{(Ia)} \geq -\frac{1}{2} \sum_{k,l,j=1}^d \|c_{kl}\|_{W^{2,\infty}} \int (|\partial_k u|^2 + |\partial_j u|^2) |\nabla u|^{p-2} \geq -M_1 \|\nabla u\|_p^p,$$

where $M_1 = d^2 \sup\{\|c_{kl}\|_{W^{2,\infty}} : 1 \leq k, l \leq d\}$. Let $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$. For (Ib) we estimate

$$\begin{aligned}
(\text{Ib}) &= -\text{Re} \sum_{j=1}^d \int \text{tr}((\partial_j C) U) (\partial_j \bar{u}) |\nabla u|^{p-2} \\
&\geq -\sum_{j=1}^d \int \left(\varepsilon |\text{tr}((\partial_j C) U)|^2 |\nabla u|^{p-2} + \frac{1}{4\varepsilon} |\partial_j \bar{u}|^2 |\nabla u|^{p-2} \right) \\
&\geq -\varepsilon' \int \text{tr}(U R_s \bar{U}) |\nabla u|^{p-2} - M_2 \|\nabla u\|_p^p,
\end{aligned}$$

where we used Corollary 2.6(a) in the last step with $\varepsilon' = 32 \varepsilon d (1 + \tan \theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty$ and $M_2 = \frac{1}{4\varepsilon}$. Note that $\varepsilon' \in (0, \varepsilon_0)$ by (17).

Next we consider the real part of (II). Note that

$$\begin{aligned}
\text{Re} \sum_{k,l,j=1}^d \int c_{kl} (\partial_j \partial_k u) \partial_l (|\nabla u|^{p-2} \partial_j \bar{u}) &= \text{Re} \sum_{k,l,j=1}^d \int c_{kl} (\partial_j \partial_k u) (\partial_l \partial_j \bar{u}) |\nabla u|^{p-2} \\
&\quad + \text{Re} \sum_{k,l,j=1}^d \int c_{kl} (\partial_j \partial_k u) (\partial_j \bar{u}) \partial_l (|\nabla u|^{p-2}) \\
&= (\text{IIa}) + (\text{IIb}).
\end{aligned}$$

For (IIa) we have

$$(\text{IIa}) = \int \text{tr}(\bar{U} (\text{Re } C) U) |\nabla u|^{p-2} = \int \text{tr}(U R_s \bar{U}) |\nabla u|^{p-2}$$

as $B_a = 0$. For (IIb) we have the following estimate

$$\begin{aligned}
(\text{IIb}) &= \text{Re} \sum_{k,l,i,j=1}^d \frac{p-2}{2} \int c_{kl} (\partial_j \partial_k u) (\partial_j \bar{u}) \left((\partial_l \partial_i u) (\partial_i \bar{u}) + (\partial_l \partial_i \bar{u}) (\partial_i u) \right) |\nabla u|^{p-4} \\
&= \frac{p-2}{2} \int \text{Re} \left((C U \nabla \bar{u}, \bar{U} \nabla \bar{u}) + (C U \nabla \bar{u}, U \nabla \bar{u}) \right) |\nabla u|^{p-4} \\
&= (p-2) \int \left((R_s \xi, \xi) - (B_s \xi, \eta) \right) |\nabla u|^{p-4},
\end{aligned}$$

where $\xi, \eta \in \mathbb{R}^d$ and $U \nabla \bar{u} = \xi + i \eta$.

In total we obtain

$$\begin{aligned}
\text{Re}(\nabla(B_p u), |\nabla u|^{p-2} \nabla u) &\geq -(M_1 + M_2) \|\nabla u\|_p^p + (1 - \varepsilon') \int \text{tr}(U R_s \bar{U}) |\nabla u|^{p-2} \\
&\quad + (p-2) \int \left((R_s \xi, \xi) - (B_s \xi, \eta) \right) |\nabla u|^{p-4} \\
&= -(M_1 + M_2) \|\nabla u\|_p^p + P,
\end{aligned}$$

where

$$P = (1 - \varepsilon') \int \text{tr}(U R_s \bar{U}) |\nabla u|^{p-2} + (p-2) \int \left((R_s \xi, \xi) - (B_s \xi, \eta) \right) |\nabla u|^{p-4}.$$

Next we will show that $P \geq 0$. Since $B_a = 0$, it follows from Lemma 2.7 that

$$\begin{aligned} (R_s \xi, \xi) + (R_s \eta, \eta) &= ((\operatorname{Re} C) U \nabla \bar{u}, U \nabla \bar{u}) \leq \operatorname{tr} (U^* (\operatorname{Re} C) U) |\nabla u|^2 \\ &= \operatorname{tr} (\bar{U} R_s U) |\nabla u|^2 = \operatorname{tr} (U R_s \bar{U}) |\nabla u|^2. \end{aligned}$$

Therefore

$$\begin{aligned} P &\geq \int \left((p-1-\varepsilon') (R_s \xi, \xi) + (1-\varepsilon') (R_s \eta, \eta) - (p-2) (B_s \xi, \eta) \right) |\nabla u|^{p-4} \\ &= \int \left((R_s \xi', \xi') + (R_s \eta', \eta') - \frac{p-2}{\sqrt{(1-\varepsilon')(p-1-\varepsilon')}} (B_s \xi', \eta') \right) |\nabla u|^{p-4}, \quad (18) \end{aligned}$$

where $\xi' = \sqrt{p-1-\varepsilon'} \xi$ and $\eta' = \sqrt{1-\varepsilon'} \eta$. If $\theta = 0$ then it follows from Lemma 2.1 that $(B_s \xi', \eta') = 0$. Therefore (18) gives

$$P \geq \int \left((R_s \xi', \xi') + (R_s \eta', \eta') \right) |\nabla u|^{p-4} \geq 0.$$

If $\theta \neq 0$ then (18) can be estimated by

$$P \geq \int \left((R_s \xi', \xi') + (R_s \eta', \eta') - 2 \cot \theta |(B_s \xi', \eta')| \right) |\nabla u|^{p-4} \geq 0,$$

where we again used Lemma 2.1. Either way we always have

$$\operatorname{Re} (\nabla (B_p u), |\nabla u|^{p-2} \nabla u) \geq -(M_1 + M_2) \|\nabla u\|_p^p$$

as claimed. \square

Proposition 4.9. *Suppose $|1 - \frac{2}{p}| < \cos \theta$ and $B_a = 0$. Then B_p is m -accretive. Furthermore $C_c^\infty(\mathbb{R}^d)$ is a core for B_p .*

Proof. We will proceed in three steps.

Step 1: We will show that $\overline{B_p}|_{D(B_p) \cap W^{1,p}(\mathbb{R}^d)}$ is m -accretive.

It follows from Propositions 4.1 and 4.7 that $B_p|_{D(B_p) \cap W^{1,p}(\mathbb{R}^d)}$ is accretive. Hence $\overline{B_p}|_{D(B_p) \cap W^{1,p}(\mathbb{R}^d)}$ is also accretive.

Next we will show that there exists a $\lambda > 0$ such that $(\lambda + B_p)(D(B_p) \cap W^{1,p}(\mathbb{R}^d))$ is dense in $L_p(\mathbb{R}^d)$. In fact we will show that there exists a $\lambda > 0$ such that $W^{1,p}(\mathbb{R}^d) \subset (\lambda + B_p)(D(B_p) \cap W^{1,p}(\mathbb{R}^d))$. Since $-\Delta$ satisfies the same conditions as those of B_p , Proposition 4.8 also applies to $-\Delta$. In particular there exists an $M' > 0$ such that

$$\operatorname{Re} (\nabla (\Delta u), |\nabla u|^{p-2} \nabla u) \geq -M' \|\nabla u\|_p^p$$

for all $u \in W^{3,p}(\mathbb{R}^d)$.

For all $n \in \mathbb{N}$ define the operator $B_{p,n}$ by

$$B_{p,n} u = B_p u - \frac{1}{n} \Delta u$$

on the domain

$$D(B_{p,n}) = W^{2,p}(\mathbb{R}^d),$$

where $\Delta = \partial_1^2 + \dots + \partial_d^2$. Note that for each $n \in \mathbb{N}$ the operator $B_{p,n}$ is strongly elliptic, which implies that $B_{p,n}$ is closed.

Let M be as in Proposition 4.8 and $\lambda = M + M' + 1$. Let $f \in W^{1,p}(\mathbb{R}^d)$. Let $n \in \mathbb{N}$. Then there exists a $u_n \in W^{2,p}(\mathbb{R}^d)$ such that $(\lambda + B_{p,n})u_n = f$. Elliptic regularity gives $u_n \in W^{3,p}(\mathbb{R}^d)$. It follows that $\nabla(B_{p,n}u_n) = \nabla(f - \lambda u_n) \in (L_p(\mathbb{R}^d))^d$ and $\nabla(B_p u_n) = \nabla(B_{p,n}u_n) + \frac{1}{n} \nabla(\Delta u_n) \in (L_p(\mathbb{R}^d))^d$. By Proposition 4.1 we have

$$(f, |u_n|^{p-2} u_n) = \lambda \|u_n\|_p^p + (B_{p,n}u_n, |u_n|^{p-2} u_n) \geq \lambda \|u_n\|_p^p \geq \|u_n\|_p^p.$$

However

$$(f, |u_n|^{p-2} u_n) \leq \|f\|_p \| |u_n|^{p-2} u_n \|_q = \|f\|_p \|u_n\|_p^{p/q}$$

by Hölder's inequality. Therefore $\|u_n\|_p^p \leq \|f\|_p \|u_n\|_p^{p/q}$, or equivalently $\|u_n\|_p \leq \|f\|_p$. Also it follows from Proposition 4.8 that

$$\begin{aligned} (\nabla f, |\nabla u_n|^{p-2} \nabla u_n) &= \lambda \|\nabla u_n\|_p^p + \operatorname{Re} \left(\nabla(B_{p,n}u_n), |\nabla u_n|^{p-2} \nabla u_n \right) \\ &= \lambda \|\nabla u_n\|_p^p + \operatorname{Re} \left(\nabla(B_p u_n), |\nabla u_n|^{p-2} \nabla u_n \right) \\ &\quad - \frac{1}{n} \operatorname{Re} \left(\nabla(\Delta u_n), |\nabla u_n|^{p-2} \nabla u_n \right) \\ &\geq (\lambda - M - M') \|\nabla u_n\|_p^p = \|\nabla u_n\|_p^p. \end{aligned}$$

Again the Hölder's inequality gives $\|\nabla u_n\|_p \leq \|\nabla f\|_p$. Hence $\|u_n\|_{W^{1,p}} \leq \|f\|_{W^{1,p}}$. In particular $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R}^d)$. Passing to a subsequence if necessary we may assume that $\{u_k\}_{k \in \mathbb{N}}$ converges weakly to a $u \in W^{1,p}(\mathbb{R}^d)$. Note that $B_{p,n}u_n = f - \lambda u_n$. Therefore $\{B_{p,n}u_n\}_{k \in \mathbb{N}}$ is bounded in $L_p(\mathbb{R}^d)$. Passing to a subsequence if necessary we again assume that $\{B_{p,k}u_k\}_{k \in \mathbb{N}}$ converges weakly to a $v \in L_p(\mathbb{R}^d)$. Then $v = f - \lambda u$. We will show that $B_p u = v$. Indeed let $\phi \in C_c^\infty(\mathbb{R}^d)$. Then $\lim_{n \rightarrow \infty} B_{p,n}^* \phi = B_p^* \phi$ strongly in $L_q(\mathbb{R}^d)$ and

$$(v, \phi) = \lim_{n \rightarrow \infty} (B_{p,n}u_n, \phi) = \lim_{n \rightarrow \infty} (u_n, B_{p,n}^* \phi) = (u, B_p^* \phi).$$

Therefore $u \in D(B_p)$ and $B_p u = v$. Hence $(\lambda + B_p)u = f$.

Step 2: We will show that $\overline{B_p|_{D(B_p) \cap W^{1,p}(\mathbb{R}^d)}} = B_p$, which implies B_p is m -accretive.

Clearly $\overline{D(B_p) \cap W^{1,p}(\mathbb{R}^d)}^{\|\cdot\|_{D(B_p)}} \subset D(B_p)$. For the reverse inclusion let $u \in D(B_p)$ and λ be defined as in Step 1. Since $(\lambda + B_p)u \in L_p(\mathbb{R}^d)$ and $\overline{B_p|_{D(B_p) \cap W^{1,p}(\mathbb{R}^d)}}$ is m -accretive, there exists a $v \in \overline{D(B_p) \cap W^{1,p}(\mathbb{R}^d)}^{\|\cdot\|_{D(B_p)}}$ such that $(\lambda + B_p)v = (\lambda + B_p)u$. Equivalently

$$(u - v, (\lambda + H_q)\phi) = 0 \tag{19}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Define $G_q = (B_p|_{C_c^\infty(\mathbb{R}^d)})^*$. Then $H_q \subset G_q$. Note that $|1 - \frac{2}{p}| < \cos \theta$ is equivalent to $|1 - \frac{2}{q}| < \cos \theta$. Furthermore C^* satisfies the same condition as those of C . Therefore all previous results apply to G_q . In particular, Proposition 4.7 gives $C_c^\infty(\mathbb{R}^d)$ is dense in $(D(G_q) \cap W^{1,q}(\mathbb{R}^d), \|\cdot\|_{D(G_q)})$ and Step 1 gives $\overline{G_q|_{D(G_q) \cap W^{1,q}(\mathbb{R}^d)}}$ is m -accretive.

Now it follows from (19) that

$$(u - v, (\lambda + G_q)\phi) = 0$$

for all $\phi \in \overline{(D(G_q) \cap W^{1,q}(\mathbb{R}^d), \|\cdot\|_{D(G_q)})}$. Since $\overline{G_q|_{D(G_q) \cap W^{1,q}(\mathbb{R}^d)}}$ is m -accretive, we must have $u = v$.

Step 3: We will show that $C_c^\infty(\mathbb{R}^d)$ is a core for B_p .

This follows immediately from Proposition 4.7 and Step 2. \square

5 The core property for A_p

Let $p \in (1, \infty)$ be such that $|1 - \frac{2}{p}| < \cos \theta$. Suppose $B_a = 0$. In Section 3, we proved that the contraction C_0 -semigroup S generated by A extends consistently to a contraction C_0 -semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$. Let $-A_p$ be the generator of $S^{(p)}$. In this section we will show that the operator A_p and B_p are in fact the same. Consequently the space of test functions $C_c^\infty(\mathbb{R}^d)$ is a core for A_p . This is the content of Theorem 1.2, which is also the main theorem of the paper.

Proposition 5.1. *Let $p \in (1, \infty)$ be such that $|1 - \frac{2}{p}| < \cos \theta$. Suppose $B_a = 0$. Then $A_p = B_p$.*

Proof. Let $u \in D(A) \cap D(A_p)$. Then

$$(A_p u, \phi) = (Au, \phi) = \mathfrak{a}(u, \phi) = (u, H_q \phi)$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. It follows that $u \in D(B_p)$ and $B_p u = A_p u$. In particular this implies that $D(A) \cap D(A_p) \subset D(B_p)$. However $D(A) \cap D(A_p)$ is a core for A_p and B_p is closed. Hence $D(A_p) \subset D(B_p)$. On the other hand note that $-A_p$ generates a contraction C_0 -semigroup. Therefore A_p is m -accretive. By Proposition 4.9 the operator B_p is also m -accretive. Hence $A_p = B_p$ as required. \square

Theorem 1.2 now follows immediately from the above proposition.

Proof of Theorem 1.2. By Proposition 4.9 the space $C_c^\infty(\mathbb{R}^d)$ is a core for B_p . Since $A_p = B_p$ by Proposition 5.1, it follows that $C_c^\infty(\mathbb{R}^d)$ is also a core for A_p . \square

6 More sufficient conditions in L_2

This section is motivated by the fact that B_2 is accretive on $W^{2,2}(\mathbb{R}^d)$ without the requirement that $B_a = 0$ (cf. Proposition 4.1). In fact more is true.

Proposition 6.1. *We have*

$$\operatorname{Re}(B_2 u, u) \geq 0$$

for all $u \in W^{1,2}(\mathbb{R}^d) \cap D(B_2)$.

Proof. Let $u \in W^{1,2}(\mathbb{R}^d) \cap D(B_2)$. Then

$$\begin{aligned} \operatorname{Re}(B_2 u, u) &= -\operatorname{Re} \sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l (c_{kl} \partial_k u)) \bar{u} = \operatorname{Re} \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} (\partial_k u) \partial_l \bar{u} \\ &= \operatorname{Re} \int_{\mathbb{R}^d} (C \nabla u, \nabla u) = \int_{\mathbb{R}^d} ((\operatorname{Re} C) \nabla u, \nabla u) \geq 0 \end{aligned}$$

as claimed. \square

Define the operator $Z = \overline{B_2|_{C_c^\infty(\mathbb{R}^d)}}$. Then Z is closed. Furthermore we have the following.

Proposition 6.2. *The operator Z is accretive and $Z = \overline{B_2|_{W^{1,2}(\mathbb{R}^d) \cap D(B_2)}}$.*

Proof. It suffices to show $Z = \overline{B_2|_{W^{1,2}(\mathbb{R}^d) \cap D(B_2)}}$. This follows immediately from Proposition 4.7. \square

From now on we drop the condition that $B_a \neq 0$. In this section we will provide many sufficient conditions for the space of test functions $C_c^\infty(\mathbb{R}^d)$ to be a core for the operator A . Define the operator L in $L_2(\mathbb{R}^d)$ as follows.

$$Lu = - \sum_{k,l=1}^d \partial_k (\overline{(B_a)_{kl}} \partial_l u) \quad (20)$$

on the domain

$$D(L) = C_c^\infty(\mathbb{R}^d).$$

Next define the operator associated with B_a as $(B_a)^{\text{op}} = L^*$, which is the dual of L . In what follows we denote $(\partial_k B_a)_{kl} = \partial_k ((B_a)_{kl})$ for all $k, l \in \{1, \dots, d\}$. Although $(B_a)^{\text{op}}$ appears to be a differential operator of second order, it is in fact a first-order differential operator. Indeed for all $u \in D((B_a)^{\text{op}})$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} ((B_a)^{\text{op}} u, \phi) &= (u, L\phi) = - \sum_{k,l=1}^d \int_{\mathbb{R}^d} u \partial_k ((B_a)_{kl} \partial_l \phi) \\ &= - \sum_{k,l=1}^d \int_{\mathbb{R}^d} u \left((\partial_k B_a)_{kl} \partial_l \phi + (B_a)_{kl} \partial_k \partial_l \phi \right) \\ &= - \sum_{k,l=1}^d \int_{\mathbb{R}^d} u (\partial_k B_a)_{kl} \partial_l \phi, \end{aligned} \quad (21)$$

where the last step follows from the anti-symmetry of B_a . Note that $(B_a)_{kl} \in W^{2,\infty}(\mathbb{R}^d)$ for all $k, l \in \{1, \dots, d\}$. Therefore it follows from (21) that $W^{1,2}(\mathbb{R}^d) \subset D((B_a)^{\text{op}})$ and

$$\begin{aligned} ((B_a)^{\text{op}} u, \phi) &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \partial_l ((\partial_k B_a)_{kl} u) \phi = \sum_{k,l=1}^d \int_{\mathbb{R}^d} ((\partial_l \partial_k B_a)_{kl} u + (\partial_k B_a)_{kl} (\partial_l u)) \phi \\ &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_k B_a)_{kl} (\partial_l u) \phi = - \sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l B_a)_{kl} (\partial_k u) \phi \end{aligned}$$

for all $u \in W^{1,2}(\mathbb{R}^d)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ since B_a is anti-symmetric. Hence

$$(B_a)^{\text{op}} u = \sum_{k,l=1}^d \partial_l ((\partial_k B_a)_{kl} u) = - \sum_{k,l=1}^d (\partial_l B_a)_{kl} \partial_k u$$

for all $u \in W^{1,2}(\mathbb{R}^d)$.

Lemma 6.3. *For all $\varepsilon > 0$ there exists an $M > 0$ such that*

$$|((B_a)^{\text{op}} u, -\Delta u)| \leq \varepsilon \int_{\mathbb{R}^d} \|(\partial_l B_a) U\|_{HS}^2 + M \|\nabla u\|_2^2$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$ and write $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$. Then

$$\begin{aligned}
|((B_a)^{\text{op}} u, -\Delta u)| &= \left| \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_l B_a)_{kl} (\partial_k u) \partial_j^2 \bar{u} \right| \\
&= \left| \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} \left((\partial_j \partial_l B_a)_{kl} \partial_k u + (\partial_l B_a)_{kl} \partial_k \partial_j u \right) \partial_j \bar{u} \right| \\
&\leq \left| \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_j \partial_l B_a)_{kl} (\partial_k u) \partial_j \bar{u} \right| + \left| \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_l B_a)_{kl} (\partial_k \partial_j u) \partial_j \bar{u} \right| \\
&= \tag{I} \qquad \qquad \qquad + \tag{II}.
\end{aligned}$$

For (I) we have

$$\left| \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_j \partial_l B_a)_{kl} (\partial_k u) \partial_j \bar{u} \right| \leq d^2 \sup_{1 \leq k, l \leq d} \|(B_a)_{kl}\|_{W^{2, \infty}} \|\nabla u\|_2^2.$$

We estimate the term (II) by

$$\begin{aligned}
\left| \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_l B_a)_{kl} (\partial_k \partial_j u) \partial_j \bar{u} \right| &= \left| \sum_{l, j=1}^d \int_{\mathbb{R}^d} ((\partial_l B_a) U)_{lj} \partial_j \bar{u} \right| \\
&\leq \varepsilon \sum_{l, j=1}^d \int_{\mathbb{R}^d} |((\partial_l B_a) U)_{lj}|^2 + \frac{d}{4\varepsilon} \|\nabla u\|_2^2 \\
&= \varepsilon \int_{\mathbb{R}^d} \|(\partial_l B_a) U\|_{HS}^2 + \frac{d}{4\varepsilon} \|\nabla u\|_2^2.
\end{aligned}$$

Hence

$$|((B_a)^{\text{op}} u, -\Delta u)| \leq \varepsilon \int_{\mathbb{R}^d} \|(\partial_l B_a) U\|_{HS}^2 + M \|\nabla u\|_2^2,$$

where

$$M = d^2 \sup_{1 \leq k, l \leq d} \|(B_a)_{kl}\|_{W^{2, \infty}} + \frac{d}{4\varepsilon}$$

as required. □

Lemma 6.4. *For all $\varepsilon > 0$ there exists an $M > 0$ such that*

$$\left| \int_{\mathbb{R}^d} \text{tr} (U B_a \bar{U}) \right| \leq \varepsilon \int_{\mathbb{R}^d} \|(\partial_l B_a) U\|_{HS}^2 + M \|\nabla u\|_2^2$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$ and write $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$. Then

$$\begin{aligned}
((B_a)^{\text{op}} u, -\Delta u) &= \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} \left(\partial_l ((B_a)_{kl} \partial_k u) \right) \partial_j^2 \bar{u} \\
&= - \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} \left(\partial_l ((\partial_j B_a)_{kl} \partial_k u + (B_a)_{kl} \partial_j \partial_k u) \right) \partial_j \bar{u} \\
&= - \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j B_a)_{kl} (\partial_k u) \partial_j \bar{u} + (\partial_j B_a)_{kl} (\partial_l \partial_k u) \partial_j \bar{u} \\
&\quad + \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (B_a)_{kl} (\partial_j \partial_k u) \partial_l \partial_j \bar{u} \\
&= - \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j B_a)_{kl} (\partial_k u) \partial_j \bar{u} + \int_{\mathbb{R}^d} \text{tr} (U B_a \bar{U}),
\end{aligned}$$

where in the last step we used $\sum_{k, l=1}^d (\partial_j B_a)_{kl} (\partial_l \partial_k u) = 0$ for all $j \in \{1, \dots, d\}$, which follows from the anti-symmetry of B_a .

Let $\varepsilon > 0$ and M be as in Lemma 6.3. Then

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} \text{tr} (U B_a \bar{U}) \right| &\leq |((B_a)^{\text{op}} u, -\Delta u)| + \left| \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j B_a)_{kl} (\partial_k u) \partial_j \bar{u} \right| \\
&\leq \varepsilon \int_{\mathbb{R}^d} \|(\partial_l B_a) U\|_{HS}^2 + (M + d^2 \|B_a\|_{W^{2, \infty}}) \|\nabla u\|_2^2,
\end{aligned}$$

where we used Lemma 6.3 in the last step. □

Lemma 6.5. *Let $u \in C_c^\infty(\mathbb{R}^d)$. Then*

$$\begin{aligned}
\text{Re} ((B_2 - B_2^*) u, -\Delta u) &= 2 \text{Im} \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j \text{Im} C)_{kl} (\partial_k u) \partial_j \bar{u} \\
&\quad + 2 \text{Im} \sum_{j=1}^d \int_{\mathbb{R}^d} \text{tr} ((\partial_j \text{Im} C) U) \partial_j \bar{u}.
\end{aligned}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$ and write $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$. Then

$$\begin{aligned}
((B_2 - B_2^*) u, -\Delta u) &= \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} \left(\partial_l ((c_{kl} - \overline{c_{lk}}) \partial_k u) \right) \partial_j^2 \bar{u} \\
&= 2i \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} \left(\partial_l ((\text{Im} C)_{kl} \partial_k u) \right) \partial_j^2 \bar{u} \\
&= -2i \sum_{k, l, j=1}^d \int_{\mathbb{R}^d} \left(\partial_l ((\partial_j \text{Im} C)_{kl} \partial_k u + (\text{Im} C)_{kl} \partial_j \partial_k u) \right) \partial_j \bar{u}
\end{aligned}$$

$$\begin{aligned}
&= -2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \left((\partial_l \partial_j \operatorname{Im} C)_{kl} (\partial_k u) + (\partial_j \operatorname{Im} C)_{kl} (\partial_l \partial_k u) \right) \partial_j \bar{u} \\
&\quad + 2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\operatorname{Im} C)_{kl} (\partial_j \partial_k u) (\partial_l \partial_j \bar{u}) \\
&= -2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j \operatorname{Im} C)_{kl} (\partial_k u) \partial_j \bar{u} - 2i \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr} ((\partial_j \operatorname{Im} C) U) \partial_j \bar{u} \\
&\quad + 2i \int_{\mathbb{R}^d} \operatorname{tr} (U (\operatorname{Im} C) \bar{U}).
\end{aligned}$$

Taking the real parts both sides gives the statement since $\operatorname{tr} (U (\operatorname{Im} C) \bar{U}) \in \mathbb{R}$. \square

Lemma 6.6. *Let $u \in C_c^\infty(\mathbb{R}^d)$. Then*

$$\begin{aligned}
\operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr} ((\partial_j C) U) \partial_j \bar{u} &= \frac{1}{2} \operatorname{Re} \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl}) (\partial_l u) \partial_k \bar{u} - 2 (\partial_k \partial_j c_{kl}) (\partial_l u) \partial_j \bar{u} \\
&\quad + \operatorname{Im} \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j \operatorname{Im} C)_{kl} (\partial_j u) \partial_k \bar{u} \\
&\quad + \operatorname{Im} \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr} ((\partial_j \operatorname{Im} C) \bar{U}) \partial_j u.
\end{aligned}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$ and write $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$. Then

$$\begin{aligned}
(B_2 u, -\Delta u) &= \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l (c_{kl} \partial_k u) \partial_j^2 u = - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \left(\partial_l ((\partial_j c_{kl}) \partial_k u + c_{kl} \partial_j \partial_k u) \right) \partial_j \bar{u} \\
&= \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} ((\partial_j c_{kl}) \partial_k u + c_{kl} \partial_j \partial_k u) \partial_l \partial_j \bar{u} \\
&= - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \left((\partial_j^2 c_{kl}) \partial_k u + (\partial_j c_{kl}) \partial_j \partial_k u + (\partial_j c_{kl}) \partial_j \partial_k u + c_{kl} \partial_j^2 \partial_k u \right) \partial_l \bar{u} \\
&= - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl}) (\partial_k u) \partial_l \bar{u} + 2 (\partial_j c_{kl}) (\partial_j \partial_k u) \partial_l \bar{u} - (\partial_j^2 u) \partial_k (c_{kl} \partial_l \bar{u}) \\
&= - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl}) (\partial_k u) \partial_l \bar{u} - 2 (\partial_j u) \left((\partial_k \partial_j c_{kl}) \partial_l \bar{u} + (\partial_j c_{kl}) (\partial_k \partial_l \bar{u}) \right) \\
&\quad + (-\Delta u, B_2^* u).
\end{aligned}$$

Hence

$$\begin{aligned} \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j c_{kl}) (\partial_l \partial_k \bar{u}) \partial_j u &= \frac{1}{2} \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl}) (\partial_k u) \partial_l \bar{u} - 2 (\partial_k \partial_j c_{kl}) (\partial_l \bar{u}) (\partial_j u) \\ &\quad + \frac{1}{2} \left((B_2 u, -\Delta u) - (-\Delta u, B_2^* u) \right). \end{aligned}$$

Replacing u by \bar{u} in the above equation and taking the real parts on both sides gives

$$\begin{aligned} \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr} ((\partial_j C) U) \partial_j \bar{u} &= \operatorname{Re} \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j c_{kl}) (\partial_l \partial_k u) \partial_j \bar{u} \\ &= \frac{1}{2} \operatorname{Re} \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl}) (\partial_k \bar{u}) \partial_l u - 2 (\partial_k \partial_j c_{kl}) (\partial_l u) (\partial_j \bar{u}) \\ &\quad + \frac{1}{2} \operatorname{Re} ((B_2 - B_2^*) \bar{u}, -\Delta \bar{u}). \end{aligned}$$

Using Lemma 6.5 we yield the result. \square

Proposition 6.7. *Suppose one of the following holds.*

- (i) *The matrix B_s has constant entries.*
- (ii) *There exist $\theta_1, \theta_2 \in [0, \frac{\pi}{2})$, $\phi \in W^{2,\infty}(\mathbb{R}^d)$ and a $d \times d$ matrix \tilde{C} with entries in $W^{2,\infty}(\mathbb{R}^d)$ such that $\theta = \theta_1 + \theta_2$, $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, \tilde{C} takes values in Σ_{θ_2} and $C = \phi \tilde{C}$. Write $\tilde{C} = \tilde{R} + i \tilde{B}$, where \tilde{R} and \tilde{B} are $d \times d$ matrix-valued functions with real-valued entries. Set $\tilde{R}_s = \frac{1}{2} (\tilde{R} + \tilde{R}^T)$. Also define $\operatorname{Re} \tilde{C} = \frac{1}{2} (\tilde{C} + (\tilde{C})^*)$. Suppose further that there exists an $h > 0$ such that*

$$\operatorname{tr} (U (\operatorname{Re} \tilde{C}) \bar{U}) \geq h \operatorname{tr} (U \tilde{R}_s \bar{U})$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.

- (iii) *There exists an $M > 0$ such that $\|(\partial_l B_a) U\|_{HS}^2 \leq M \operatorname{tr} (U R_s \bar{U})$ for all $l \in \{1, \dots, d\}$ and $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.*

Then Z is m -accretive.

Proof. By Proposition 6.2 we have that $D(-\Delta) = W^{2,2}(\mathbb{R}^d) \subset D(Z)$. We will show that there exists a $\beta \in \mathbb{R}$ such that

$$\operatorname{Re} (Zu, -\Delta u) \geq -\beta \|\nabla u\|_2^2 \quad (22)$$

for all $u \in D(-\Delta) = W^{2,2}(\mathbb{R}^d)$. It then follows from [Ouh05, Theorem 1.50] that Z is m -accretive. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{2,2}(\mathbb{R}^d)$ and is a core for Z , it suffices to show that (22) holds for all $u \in C_c^\infty(\mathbb{R}^d)$.

Let $u \in C_c^\infty(\mathbb{R}^d)$ and $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$. Using integration by parts we obtain

$$\begin{aligned} (Zu, -\Delta u) &= \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l (c_{kl} \partial_k u)) \partial_j^2 \bar{u} = - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \left(\partial_l ((\partial_j c_{kl}) (\partial_k u) + c_{kl} (\partial_j \partial_k u)) \right) \partial_j \bar{u} \\ &= - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j c_{kl}) (\partial_k u) \partial_j \bar{u} + (\partial_j c_{kl}) (\partial_l \partial_k u) \partial_j \bar{u} - c_{kl} (\partial_j \partial_k u) \partial_l \partial_j \bar{u} \\ &= - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j c_{kl}) (\partial_k u) \partial_j \bar{u} - \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr} ((\partial_j C) U) \partial_j \bar{u} + \int_{\mathbb{R}^d} \operatorname{tr} (U C \bar{U}). \end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Re}(Zu, -\Delta u) &= -\operatorname{Re} \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l \partial_j c_{kl}) (\partial_k u) \partial_j \bar{u} - \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr} ((\partial_j C) U) \partial_j \bar{u} \\
&\quad + \int_{\mathbb{R}^d} \operatorname{tr} (U (\operatorname{Re} C) \bar{U}) \\
&= \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}$$

The estimate for (I) is straightforward as

$$\text{(I)} \geq - \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} |(\partial_l \partial_j c_{kl}) (\partial_k u) \partial_j \bar{u}| \geq -M_1 \|\nabla u\|_2^2, \quad (23)$$

where $M_1 = d^2 \sup_{1 \leq k,l \leq d} \|c_{kl}\|_{W^{2,\infty}}$. The estimates for (II) and (III) are more involved. We consider three cases according to the three conditions (i), (ii) and (iii) imposed above.

Case 1: Suppose (i) holds.

Since $U = U^T$ and $R_a = -R_a^T$, we have

$$\operatorname{tr} ((\partial_j R_a) U) = \operatorname{tr} (U^T (\partial_j R_a)^T) = -\operatorname{tr} (U (\partial_j R_a)) = -\operatorname{tr} ((\partial_j R_a) U).$$

Therefore $\operatorname{tr} ((\partial_j R_a) U) = 0$. This implies

$$\operatorname{tr} ((\partial_j \operatorname{Im} C) U) = \operatorname{tr} ((\partial_j B_s) U) - i \operatorname{tr} ((\partial_j R_a) U) = \operatorname{tr} ((\partial_j B_s) U) = 0,$$

where the last equality follows from the hypothesis. Using Lemma 6.6 we obtain that

$$\text{(II)} = \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \operatorname{Re} \left(\frac{1}{2} (\partial_j^2 c_{kl}) (\partial_l u) \partial_k \bar{u} - (\partial_k \partial_j c_{kl}) (\partial_l u) \partial_j \bar{u} \right) + \operatorname{Im} \left((\partial_l \partial_j \operatorname{Im} C)_{kl} (\partial_j u) \partial_k \bar{u} \right).$$

Consequently

$$\text{(II)} \geq -M_2 \|\nabla u\|_2^2,$$

where $M_2 = 3 d^2 \sup_{1 \leq k,l \leq d} \|c_{kl}\|_{W^{2,\infty}}$. Note that (III) ≥ 0 . Hence

$$\operatorname{Re}(Zu, -\Delta u) \geq -(M_1 + M_2) \|\nabla u\|_2^2.$$

Case 2: Suppose (ii) holds.

We first consider (II). We have

$$\begin{aligned}
\text{(II)} &= -\operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr} (\partial_j (\phi \tilde{C}) U) \partial_j \bar{u} \\
&= -\operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} (\partial_j \phi) \operatorname{tr} (\tilde{C} U) \partial_j \bar{u} - \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \phi \operatorname{tr} ((\partial_j \tilde{C}) U) \partial_j \bar{u} \\
&= \quad \quad \quad \text{(IIa)} \quad \quad \quad + \quad \quad \quad \text{(IIb)}.
\end{aligned}$$

Let

$$M_3 = 64 d (1 + \tan \theta_1)^2 (1 + \tan \theta_2)^2 \|\tilde{R}_s\|_\infty \sup_{1 \leq j \leq d} \|\partial_j^2 \phi\|_\infty$$

and

$$M_4 = 32 d^2 (1 + \tan \theta_1) (1 + \tan \theta_2)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 \tilde{C}\|_\infty.$$

Let

$$\varepsilon = \frac{(1 - \tan \theta_1 \tan \theta_2) h}{4 (M_3 \vee M_4 \vee 1)}.$$

Note that $\varepsilon > 0$ as $1 - \tan \theta_1 \tan \theta_2 > 0$. Indeed, if $\tan \theta = 0$ then $\theta = \theta_1 = \theta_2 = 0$, which implies $1 - \tan \theta_1 \tan \theta_2 = 1 > 0$. If $\tan \theta > 0$ then $1 - \tan \theta_1 \tan \theta_2 = \frac{\tan \theta_1 + \tan \theta_2}{\tan \theta} > 0$.

For (IIa) we estimate

$$(IIa) \geq -\varepsilon \int_{\mathbb{R}^d} \left(\sum_{j=1}^d |\partial_j \phi|^2 \right) |\operatorname{tr}(\tilde{C} U)|^2 - \frac{1}{4\varepsilon} \|\nabla u\|_2^2.$$

Note that

$$|\partial_j \phi|^2 \leq 4 (1 + \tan \theta_1)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 \phi\|_\infty \operatorname{Re} \phi$$

for all $j \in \{1, \dots, d\}$ by Lemma 2.3. Moreover,

$$|\operatorname{tr}(\tilde{C} U)|^2 \leq d \|\tilde{C} U\|_{HS}^2 \leq 16 d (1 + \tan \theta_2)^2 \|\tilde{R}_s\|_\infty \operatorname{tr}(U \tilde{R}_s \overline{U}),$$

where we used Lemma 2.12 in the last step. Consequently

$$\begin{aligned} (IIa) &\geq -\varepsilon M_3 \int_{\mathbb{R}^d} (\operatorname{Re} \phi) \operatorname{tr}(U \tilde{R}_s \overline{U}) - \frac{1}{4\varepsilon} \|\nabla u\|_2^2 \\ &\geq -\frac{(1 - \tan \theta_1 \tan \theta_2) h}{4} \int_{\mathbb{R}^d} (\operatorname{Re} \phi) \operatorname{tr}(U \tilde{R}_s \overline{U}) - \frac{1}{4\varepsilon} \|\nabla u\|_2^2. \end{aligned} \quad (24)$$

For (IIb) we estimate as follows. Since $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, we have

$$|\phi| \leq |\operatorname{Re} \phi| + |\operatorname{Im} \phi| \leq (1 + \tan \theta_1) \operatorname{Re} \phi.$$

Therefore

$$\begin{aligned} (IIb) &\geq -\sum_{j=1}^d \int_{\mathbb{R}^d} |\phi| |\operatorname{tr}((\partial_j \tilde{C}) U)| |\partial_j \overline{u}| \\ &\geq -(1 + \tan \theta_1) \sum_{j=1}^d \int_{\mathbb{R}^d} (\operatorname{Re} \phi) \left(\varepsilon |\operatorname{tr}((\partial_j \tilde{C}) U)|^2 + \frac{1}{4\varepsilon} |\partial_j \overline{u}|^2 \right) \\ &\geq -\varepsilon (1 + \tan \theta_1) \sum_{j=1}^d \int_{\mathbb{R}^d} (\operatorname{Re} \phi) |\operatorname{tr}((\partial_j \tilde{C}) U)|^2 - \frac{(1 + \tan \theta_1) \|\phi\|_\infty}{4\varepsilon} \|\nabla u\|_2^2 \\ &\geq -\varepsilon M_4 \int_{\mathbb{R}^d} (\operatorname{Re} \phi) \operatorname{tr}(U \tilde{R}_s \overline{U}) - \frac{(1 + \tan \theta_1) \|\phi\|_\infty}{4\varepsilon} \|\nabla u\|_2^2 \\ &\geq -\frac{(1 - \tan \theta_1 \tan \theta_2) h}{4} \int_{\mathbb{R}^d} (\operatorname{Re} \phi) \operatorname{tr}(U \tilde{R}_s \overline{U}) - \frac{(1 + \tan \theta_1) \|\phi\|_\infty}{4\varepsilon} \|\nabla u\|_2^2, \end{aligned} \quad (25)$$

where we used Corollary 2.6(a) in the fourth step.

On the other hand, estimating (III) gives

$$(III) = \int_{\mathbb{R}^d} \operatorname{tr} (U (\operatorname{Re} (\phi \tilde{C})) \overline{U}) = \int_{\mathbb{R}^d} (\operatorname{Re} \phi) \operatorname{tr} (U (\operatorname{Re} \tilde{C}) \overline{U}) - (\operatorname{Im} \phi) \operatorname{tr} (U (\operatorname{Im} \tilde{C}) \overline{U}).$$

Since $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, we have $|\operatorname{Im} \phi| \leq (\tan \theta_1) \operatorname{Re} \phi$. Also as \tilde{C} takes values in Σ_{θ_2} , we deduce that $|\operatorname{Im} (\tilde{C} U e_j, U e_j)| \leq (\tan \theta_2) \operatorname{Re} (\tilde{C} U e_j, U e_j)$, which in turns implies that $|\operatorname{tr} (U (\operatorname{Im} \tilde{C}) \overline{U})| \leq (\tan \theta_2) \operatorname{tr} (U (\operatorname{Re} \tilde{C}) \overline{U})$. Therefore

$$\begin{aligned} (III) &\geq \int_{\mathbb{R}^d} (1 - \tan \theta_1 \tan \theta_2) (\operatorname{Re} \phi) \operatorname{tr} (U (\operatorname{Re} \tilde{C}) \overline{U}) \\ &\geq \int_{\mathbb{R}^d} (1 - \tan \theta_1 \tan \theta_2) h (\operatorname{Re} \phi) \operatorname{tr} (U \tilde{R}_s \overline{U}) \end{aligned} \quad (26)$$

by the hypothesis. Hence by (23), (24), (25) and (26) we have

$$\begin{aligned} \operatorname{Re} (Zu, -\Delta u) &\geq \frac{(1 - \tan \theta_1 \tan \theta_2) h}{2} \int_{\mathbb{R}^d} (\operatorname{Re} \phi) \operatorname{tr} (U \tilde{R}_s \overline{U}) \\ &\quad - \left(M_1 + \frac{1 + (1 + \tan \theta_1) \|\phi\|_\infty}{4\varepsilon} \right) \|\nabla u\|_2^2 \\ &\geq - \left(M_1 + \frac{1 + (1 + \tan \theta_1) \|\phi\|_\infty}{4\varepsilon} \right) \|\nabla u\|_2^2. \end{aligned}$$

Case 3: Suppose (iii) holds.

Let $\varepsilon_1 = \frac{1}{2M}$ and M' be corresponding to ε_1 as in Lemma 6.4. Then

$$\begin{aligned} (III) &= \int_{\mathbb{R}^d} \operatorname{tr} (U R_s \overline{U}) + i \int_{\mathbb{R}^d} \operatorname{tr} (U B_a \overline{U}) \\ &\geq \int_{\mathbb{R}^d} \operatorname{tr} (U R_s \overline{U}) - \varepsilon_1 \int_{\mathbb{R}^d} \|(\partial_l B_a) U\|_{HS}^2 - M' \|\nabla u\|_2^2 \\ &\geq \frac{1}{2} \int_{\mathbb{R}^d} \operatorname{tr} (U R_s \overline{U}) - M' \|\nabla u\|_2^2 \end{aligned}$$

since $\|(\partial_l B_a) U\|_{HS}^2 \leq M \operatorname{tr} (U R_s \overline{U})$ by hypothesis.

Let $\varepsilon_2 = \frac{1}{4dM''}$, where M'' is the constant as in Corollary 2.6(a). Then

$$(II) \geq -\varepsilon_2 \sum_{j=1}^d \int_{\mathbb{R}^d} |\operatorname{tr} ((\partial_j C) U)|^2 - \frac{1}{4\varepsilon_2} \|\nabla u\|_2^2 \geq -\frac{1}{4} \int_{\mathbb{R}^d} \operatorname{tr} (U R_s \overline{U}) - \frac{1}{4\varepsilon_2} \|\nabla u\|_2^2,$$

where we used Corollary 2.6(a) in the last step. Hence

$$\operatorname{Re} (Zu, -\Delta u) \geq \frac{1}{4} \int_{\mathbb{R}^d} \operatorname{tr} (U R_s \overline{U}) - \left(\frac{1}{4\varepsilon_2} + M_1 + M' \right) \|\nabla u\|_2^2.$$

The proof is complete. \square

We emphasise that it is not known yet whether B_2 is accretive if $B_a \neq 0$. The following theorem is of main interest and will be used extensively.

Theorem 6.8. *Suppose one of the following holds.*

- (i) The matrix B_s has constant entries.
- (ii) There exist $\theta_1, \theta_2 \in [0, \frac{\pi}{2})$, $\phi \in W^{2,\infty}(\mathbb{R}^d)$ and a $d \times d$ matrix \tilde{C} with entries in $W^{2,\infty}(\mathbb{R}^d)$ such that $\theta = \theta_1 + \theta_2$, $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, \tilde{C} takes values in Σ_{θ_2} and $C = \phi \tilde{C}$. Write $\tilde{C} = \tilde{R} + i\tilde{B}$, where \tilde{R} and \tilde{B} are $d \times d$ matrix-valued functions with real-valued entries. Set $\tilde{R}_s = \frac{1}{2}(\tilde{R} + \tilde{R}^T)$. Also define $\text{Re } \tilde{C} = \frac{1}{2}(\tilde{C} + (\tilde{C})^*)$. Suppose further that there exists an $h > 0$ such that

$$\text{tr}(U(\text{Re } \tilde{C})\overline{U}) \geq h \text{tr}(U\tilde{R}_s\overline{U})$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.

- (iii) There exists an $M > 0$ such that $\|(\partial_l B_a)U\|_{HS}^2 \leq M \text{tr}(U R_s \overline{U})$ for all $l \in \{1, \dots, d\}$ and $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.

Then $A = B_2 = Z$. Moreover, $C_c^\infty(\mathbb{R}^d)$ is a core for A .

Proof. By Proposition 6.7 the operator Z is m -accretive. We will show that $Z = B_2$. Clearly $Z \subset B_2$. For the reverse inclusion let $u \in D(B_2)$. Then $(I + B_2)u \in L_2(\mathbb{R}^d)$. Since Z is m -accretive, there exists a $v \in D(Z)$ such that $(I + Z)v = (I + B_2)u$. But B_2 is an extension of Z . Therefore $(I + B_2)v = (I + B_2)u$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$0 = ((I + B_2)(u - v), \phi) = (u - v, (I + H_2)\phi),$$

where H_2 is defined by (4). But H_2 satisfies the same criteria as those of $B_2|_{C_c^\infty(\mathbb{R}^d)}$. Therefore analogous arguments give that $\overline{H_2}$ is also m -accretive. Consequently $u = v$. Hence $Z = B_2$. It follows that B_2 is m -accretive. In particular B_2 is accretive. Note that A is m -accretive and $A \subset B_2$. Therefore we must have $A = B_2 = Z$. Moreover, since $C_c^\infty(\mathbb{R}^d)$ is a core for Z , it is also a core for A . \square

The next proposition provides three easy criteria to verify Condition (iii) in Theorem 6.8.

Proposition 6.9. Suppose C satisfies one of the following.

- (a) There exists an $r \in \mathbb{R} \setminus \{0\}$ such that $R_s + ir \partial_l B_a \geq 0$ for all $l \in \{1, \dots, d\}$.
- (b) The matrices R_s and $\partial_l B_a$ commute for all $l \in \{1, \dots, d\}$.
- (c) There exist a real-valued function $\phi \in W^{2,\infty}(\mathbb{R}^d)$ which satisfies $\phi \geq 0$ and a $d \times d$ matrix \tilde{C} which has constant entries and takes values in Σ_θ such that $C = \phi \tilde{C}$.

Then there exists an $M > 0$ such that $\|(\partial_l B_a)U\|_{HS}^2 \leq M \text{tr}(U R_s \overline{U})$ for all $l \in \{1, \dots, d\}$ and $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.

Proof. Let $l \in \{1, \dots, d\}$. Let $u \in C_c^\infty(\mathbb{R}^d)$ and $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$.

We first deal with (a) and (b). Set $P = \sqrt{U \overline{U}} \geq 0$. Let V be a unitary matrix such that $P = V D_P V^*$, where D_P is a positive diagonal matrix. Then

$$\begin{aligned} \|(\partial_l B_a)U\|_{HS}^2 &= -\text{tr}(U^*(\partial_l B_a)^2 U) = -\text{tr}((\partial_l B_a)^2 P^2) = -\text{tr}((\partial_l B_a)^2 V D_P^2 V^*) \\ &= -\text{tr}(V^*(\partial_l B_a)^2 V D_P^2) = \sum_{k=1}^d |(V^*(\partial_l B_a)^2 V)_{kk}| |(D_P)_{kk}|^2. \end{aligned}$$

We consider two cases.

Case 1: Suppose (a) holds.

Then $|((\partial_l B_a) \xi, \xi)| \leq \frac{1}{|r|} (R_s \xi, \xi)$ for all $\xi \in \mathbb{C}^d$. By Lemma 2.10 we have $\|(\partial_l B_a) \xi\|^2 \leq \frac{4}{r^2} \|R_s\|_\infty (R_s \xi, \xi)$ for all $\xi \in \mathbb{C}^d$. In particular $\|(\partial_l B_a) V e_k\|^2 \leq \frac{4}{r^2} \|R_s\|_\infty (V^* R_s V)_{kk}$ for all $k \in \{1, \dots, d\}$. It follows that

$$\begin{aligned} \|(\partial_l B_a) U\|_{HS}^2 &\leq \frac{4}{r^2} \|R_s\|_\infty \sum_{k=1}^d (V^* R_s V)_{kk} |(D_P)_{kk}|^2 = \frac{4}{r^2} \|R_s\|_\infty \operatorname{tr} (V^* R_s V D_P^2) \\ &= \frac{4}{r^2} \|R_s\|_\infty \operatorname{tr} (R_s P^2) = \frac{4}{r^2} \|R_s\|_\infty \operatorname{tr} (U^* R_s U) \\ &= \frac{4}{r^2} \|R_s\|_\infty \operatorname{tr} (U R_s U^*) = \frac{4}{r^2} \|R_s\|_\infty \operatorname{tr} (U R_s \overline{U}), \end{aligned}$$

where the last equality follows from the fact that $U = U^T$.

Case 2: Suppose (b) holds.

Let W be a unitary matrix such that $\partial_l B_a = W D W^*$, where D is diagonal. Therefore

$$|D_{kk}|^2 = |(W^* (\partial_l B_a) W)_{kk}|^2 \leq 2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty (W^* R_s W)_{kk}$$

for all $k \in \{1, \dots, d\}$ by Lemma 2.9. Since R_s and $\partial_l B_a$ commute, we may assume without loss of generality that the matrix W also diagonalises R_s . It follows that

$$\begin{aligned} |(V^* (\partial_l B_a)^2 V)_{kk}| &= |(V^* W D^2 W^* V)_{kk}| = |((W^* V)^* D^2 W^* V)_{kk}| \\ &= \sum_{j=1}^d ((W^* V)^*)_{kj} |D_{jj}|^2 (W^* V)_{jk} \\ &\leq 2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty \sum_{j=1}^d ((W^* V)^*)_{kj} (W^* R_s W)_{jj} (W^* V)_{jk} \\ &= 2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty (V^* R_s V)_{kk} \end{aligned}$$

for all $k \in \{1, \dots, d\}$. Hence

$$\begin{aligned} \|(\partial_l B_a) U\|_{HS}^2 &\leq 2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty \sum_{k=1}^d (V^* R_s V)_{kk} |(D_P)_{kk}|^2 \\ &= 2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty \operatorname{tr} (V^* R_s V D_P^2) = 2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty \operatorname{tr} (R_s P^2) \\ &= 2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty \operatorname{tr} (U R_s \overline{U}). \end{aligned}$$

This completes the proof of the proposition under the assumptions (a) and (b).

Next we turn to (c). Suppose (c) holds. Write $\tilde{C} = \tilde{R} + i \tilde{B}$. Set $\tilde{R}_s = \frac{1}{2} (\tilde{R} + \tilde{R}^T)$ and $\tilde{B}_a = \frac{1}{2} (\tilde{B} - \tilde{B}^T)$. Since ϕ is real-valued, we have $R_s = \phi \tilde{R}_s$ and $B_a = \phi \tilde{B}_a$. Applying Lemma 2.2 to ϕ we obtain $(\partial_l \phi)^2 \leq 2 \|\phi\|_{W^{2,\infty}} \phi$. By Lemmas 2.8 and 2.10 we also have

$\|\tilde{B}_a \xi\|^2 \leq 4 \|\tilde{R}_s\|_\infty (\tilde{R}_s \xi, \xi)$ for all $\xi \in \mathbb{C}^d$. Therefore

$$\begin{aligned} \|(\partial_l B_a) U\|_{HS}^2 &= \sum_{j=1}^d \|(\partial_l B_a) U e_j\|_2^2 = (\partial_l \phi)^2 \sum_{j=1}^d \|\tilde{B}_a U e_j\|_2^2 \\ &\leq 8 \|\phi\|_{W^{2,\infty}} \|\tilde{R}_s\|_\infty \phi \sum_{j=1}^d (\tilde{R}_s U e_j, U e_j) = 8 \|\phi\|_{W^{2,\infty}} \|\tilde{R}_s\|_\infty \operatorname{tr}(U R_s \bar{U}). \end{aligned}$$

The proof is complete. \square

Our next aim is to show that if $D(A) \subset W^{1,2}(\mathbb{R}^d)$, then $C_c^\infty(\mathbb{R}^d)$ is a core for A .

Lemma 6.10. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then*

$$\sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} \eta (\partial_k u) \partial_l \bar{\phi} = \left(\eta A u - \sum_{k,l=1}^d c_{kl} (\partial_k u) \partial_l \eta, \phi \right)$$

for all $u \in D(A)$ and $\eta, \phi \in C_c^\infty(\mathbb{R}^d)$.

Proof. Let $u \in D(A)$ and $\eta, \phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$\begin{aligned} (\eta A u, \phi) &= (A u, \bar{\eta} \phi) = \mathfrak{a}(u, \bar{\eta} \phi) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} (\partial_k u) \partial_l (\eta \bar{\phi}) \\ &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} (\partial_k u) (\partial_l \eta) \bar{\phi} + \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} (\partial_k u) \eta \partial_l \bar{\phi}. \end{aligned}$$

Next we rearrange the terms to derive the lemma. \square

Recall that J_n is the usual mollifier with respect to a suitable function in $C_c^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$.

Proposition 6.11. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then $C_c^\infty(\mathbb{R}^d)$ is a core for A if and only if $\lim_{n \rightarrow \infty} A(J_n * u) = A u$ in $L_2(\mathbb{R}^d)$ for all $u \in D(A)$.*

Proof. (\implies) It is well-known that $\lim_{n \rightarrow \infty} J_n * (A u) = A u$ in $L_2(\mathbb{R}^d)$. Therefore it suffices to show that $\lim_{n \rightarrow \infty} \|A(J_n * u) - J_n * (A u)\|_2 = 0$.

By a similar calculation as in (11) we yield

$$A(J_n * u) - J_n * A u = T_n u \tag{27}$$

for all $n \in \mathbb{N}$ and $u \in C_c^\infty(\mathbb{R}^d)$, where the bounded operator $T_n : W^{1,2}(\mathbb{R}^d) \longrightarrow L_2(\mathbb{R}^d)$ is defined by (6). Let $n \in \mathbb{N}$ and $u \in D(A)$. Since $C_c^\infty(\mathbb{R}^d)$ is a core for $D(A)$, there exists a sequence $\{\phi_j\}_{j \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that

$$\lim_{j \rightarrow \infty} \phi_j = u \tag{28}$$

in $D(A)$. By hypothesis $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Therefore the inclusion $D(A) \hookrightarrow W^{1,2}(\mathbb{R}^d)$ is continuous. It follows from (28) that $\lim_{j \rightarrow \infty} \phi_j = u$ in $W^{1,2}(\mathbb{R}^d)$. Recall that the operator T_n is bounded. As a consequence $\lim_{j \rightarrow \infty} T_n \phi_j = T_n u$ in $L_2(\mathbb{R}^d)$. We also derive from (28)

that $\lim_{j \rightarrow \infty} J_n * \phi_j = J_n * u$ in $L_2(\mathbb{R}^d)$ and $\lim_{j \rightarrow \infty} J_n * (A\phi_j) = J_n * (Au)$ in $L_2(\mathbb{R}^d)$. Therefore (27) gives

$$\lim_{j \rightarrow \infty} A(J_n * \phi_j) = \lim_{j \rightarrow \infty} (T_n \phi_j + J_n * (A\phi_j)) = T_n u + J_n * (Au)$$

in $L_2(\mathbb{R}^d)$. Since T_n is bounded, it is also closed. Hence $J_n * u \in D(A)$ and $A(J_n * u) = T_n u + J_n * (Au)$. That is,

$$A(J_n * u) - J_n * Au = T_n u \quad (29)$$

also holds for all $n \in \mathbb{N}$ and $u \in D(A)$.

Let $\psi \in W^{2,2}(\mathbb{R}^d)$. Then $\lim_{n \rightarrow \infty} J_n * \psi = \psi$ in $W^{2,2}(\mathbb{R}^d)$. Consequently $\lim_{n \rightarrow \infty} A(J_n * \psi) = A\psi$ in $L_2(\mathbb{R}^d)$. Also $\lim_{n \rightarrow \infty} J_n * (A\psi) = A\psi$ in $L_2(\mathbb{R}^d)$. Therefore it follows from (29) that $\lim_{n \rightarrow \infty} \|T_n u\|_2 = 0$. This is for all $\psi \in W^{2,2}(\mathbb{R}^d)$. Since $W^{2,2}(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d)$ and $\{T_n\}_{n \in \mathbb{N}}$ is bounded by Lemma 4.4, we deduce that $\lim_{n \rightarrow \infty} \|T_n u\|_2 = 0$ for all $u \in W^{1,2}(\mathbb{R}^d)$. In particular $\lim_{n \rightarrow \infty} \|T_n u\|_2 = 0$ for all $u \in D(A)$ as $D(A) \subset W^{1,2}(\mathbb{R}^d)$ by hypothesis.

(\Leftarrow) Let $\tau \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \leq \tau \leq 1$, $\tau|_{B_1(0)} = 1$ and $\text{supp } \tau \subset B_2(0)$. Define $\tau_n(x) = \tau(n^{-1}x)$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Let $u \in D(A)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Then $u \in W^{1,2}(\mathbb{R}^d)$ and hence $\tau_n u \in W^{1,2}(\mathbb{R}^d)$. Moreover

$$\mathfrak{a}(\tau_n u, \phi) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} \partial_k(\tau_n u) \partial_l \bar{\phi} = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} ((\partial_k \tau_n) u + \tau_n \partial_k u) \partial_l \bar{\phi} = (f_n, \phi),$$

where

$$f_n = (A\tau_n) u + \tau_n Au - \sum_{k,l=1}^d c_{kl} (\partial_k \tau_n) \partial_l u - \sum_{k,l=1}^d c_{kl} (\partial_l \tau_n) \partial_k u$$

and we used Lemma 6.10 in the last equality. Since $f_n \in L_2(\mathbb{R}^d)$, we have $\tau_n u \in D(A)$ and $A(\tau_n u) = f_n$. Next we will show that $\lim_{n \rightarrow \infty} f_n = Au$ in $L_2(\mathbb{R}^d)$. Clearly $\lim_{n \rightarrow \infty} \tau_n Au = Au$ in $L_2(\mathbb{R}^d)$. Note that

$$\begin{aligned} \|(A\tau_n) u\|_2 &= \left\| - \sum_{k,l=1}^d (\partial_l (c_{kl} \partial_k \tau_n)) u \right\|_2 = \left\| \sum_{k,l=1}^d \left((\partial_l c_{kl}) \partial_k \tau_n + c_{kl} \partial_l \partial_k \tau_n \right) u \right\|_2 \\ &\leq \sum_{k,l=1}^d \|c_{kl}\|_{W^{2,\infty}} \left(\frac{1}{n} \|\partial_k \tau\|_\infty + \frac{1}{n^2} \|\partial_l \partial_k \tau\|_\infty \right) \|u\|_2. \end{aligned}$$

Similarly

$$\left\| \sum_{k,l=1}^d c_{kl} (\partial_k \tau_n) \partial_l u \right\|_2 \leq \frac{1}{n} \sum_{k,l=1}^d \|c_{kl}\|_\infty \|\partial_k \tau\|_\infty \|\partial_l u\|_2$$

and

$$\left\| \sum_{k,l=1}^d c_{kl} (\partial_l \tau_n) \partial_k u \right\|_2 \leq \frac{1}{n} \sum_{k,l=1}^d \|c_{kl}\|_\infty \|\partial_l \tau\|_\infty \|\partial_k u\|_2.$$

It follows that these three terms go to 0 in $L_2(\mathbb{R}^d)$ as n tends to infinity. Hence

$$\lim_{n \rightarrow \infty} \|A(\tau_n u) - Au\|_2 = 0. \quad (30)$$

Finally we will show that $C_c^\infty(\mathbb{R}^d)$ is a core for A . Let $u \in D(A)$. The hypothesis gives

$$\lim_{k \rightarrow \infty} \|A(J_k * (\tau_n u)) - A(\tau_n u)\|_2 = 0 \quad (31)$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. By (30) we can choose an $n \in \mathbb{N}$ such that $\|A(\tau_n u) - Au\|_2 < \frac{\varepsilon}{2}$. Next we use (31) to choose a $k \in \mathbb{N}$ such that $\|A(J_k * (\tau_n u)) - A(\tau_n u)\|_2 < \frac{\varepsilon}{2}$. Then

$$\|A(J_k * (\tau_n u)) - Au\|_2 \leq \|A(J_k * (\tau_n u)) - A(\tau_n u)\|_2 + \|A(\tau_n u) - Au\|_2 < \varepsilon.$$

Note that $J_k * (\tau_n u) \in C_c^\infty(\mathbb{R}^d)$. Hence $C_c^\infty(\mathbb{R}^d)$ is indeed a core for A . \square

Let $\delta \in (0, 1)$. Define

$$C_\delta = (R_s + i\delta B_a) + i(B_s - iR_a).$$

Lemma 6.12. *The matrix C_δ takes values in Σ_ψ , where $\psi \in [0, \frac{\pi}{2})$ is such that $\tan \psi = \frac{1}{\delta} \tan \theta$.*

Proof. Let $\xi \in \mathbb{C}^d$. Then

$$\begin{aligned} |((\operatorname{Im} C_\delta) \xi, \xi)| &= |((\operatorname{Im} C) \xi, \xi)| \leq \tan \theta ((\operatorname{Re} C) \xi, \xi) = \frac{1}{\delta} \tan \theta ((\delta R_s + i\delta B_a) \xi, \xi) \\ &\leq \frac{1}{\delta} \tan \theta ((R_s + i\delta B_a) \xi, \xi) = \frac{1}{\delta} \tan \theta ((\operatorname{Re} C_\delta) \xi, \xi) \end{aligned}$$

since C takes values in Σ_θ and $(R_s \xi, \xi) \geq 0$ by Lemma 2.8. The statement now follows. \square

Define the form

$$\mathfrak{a}_{0,\delta}(u, v) = \int_{\mathbb{R}^d} (C_\delta \nabla u, \nabla v)$$

on the domain $D(\mathfrak{a}_{0,\delta}) = C_c^\infty(\mathbb{R}^d)$. Then by the same analysis as in Section 1, the form $\mathfrak{a}_{0,\delta}$ is closable. Let A_δ be the operator associated with the closure of $\mathfrak{a}_{0,\delta}$. Then we also have that $W^{2,2}(\mathbb{R}^d) \subset D(A_\delta)$ and

$$A_\delta u = - \sum_{k,l=1}^d \partial_l ((C_\delta)_{kl} \partial_k u)$$

for all $u \in W^{2,2}(\mathbb{R}^d)$. Define

$$H_\delta = - \sum_{k,l=1}^d \partial_k (\overline{(C_\delta)_{kl}} \partial_l u)$$

on the domain $D(H_\delta) = C_c^\infty(\mathbb{R}^d)$. Then we have the following.

Proposition 6.13. *The space $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ . Furthermore $A_\delta = (H_\delta)^*$.*

Proof. We note that

$$\operatorname{tr} (U (\operatorname{Re} C_\delta) \overline{U}) = (1 - \delta) \operatorname{tr} (U R_s \overline{U}) + \delta \operatorname{tr} (U (\operatorname{Re} C) \overline{U}) \geq (1 - \delta) \operatorname{tr} (U R_s \overline{U})$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \leq k, l \leq d}$. That is, C_δ satisfies Condition (ii) in Theorem 6.8. Hence $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ and $A_\delta = (H_\delta)^*$ by Theorem 6.8. \square

Lemma 6.14. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then $D(A) \subset D(A_\delta) \cap D((B_a)^{\text{op}})$ and*

$$Au = A_\delta u + i(1 - \delta) (B_a)^{\text{op}} u$$

for all $u \in D(A)$.

Proof. Recall that the operators H_2 and L are defined by (4) and (20) respectively. First note that $D(A) \subset W^{1,2}(\mathbb{R}^d) \subset D((B_a)^{\text{op}})$. Moreover, the condition $D(A) \subset W^{1,2}(\mathbb{R}^d)$ implies that

$$(u, H_2 \phi) = - \int_{\mathbb{R}^d} u \partial_k (c_{kl} \partial_l \bar{\phi}) = \int_{\mathbb{R}^d} c_{kl} (\partial_k u) \partial_l \bar{\phi} = \mathfrak{a}(u, \phi) = (Au, \phi)$$

for all $u \in D(A)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, where we used integration by parts in the second step. Since $Au \in L_2(\mathbb{R}^d)$, we conclude that $u \in D(B_2)$ and

$$B_2 u = Au \tag{32}$$

for all $u \in D(A)$. Therefore we also have $D(A) \subset D(B_2)$.

Next let $u \in D(A)$. Then

$$\begin{aligned} (u, H_\delta \phi) &= (u, H_2 \phi) - i(1 - \delta) (u, L \phi) = (B_2 u, \phi) - i(1 - \delta) ((B_a)^{\text{op}} u, \phi) \\ &= (B_2 u - i(1 - \delta) (B_a)^{\text{op}} u, \phi) \end{aligned}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. Note that $B_2 u - i(1 - \delta) (B_a)^{\text{op}} u \in L_2(\mathbb{R}^d)$. Hence $u \in D(A_\delta)$ and

$$A_\delta u = B_2 u - i(1 - \delta) (B_a)^{\text{op}} u = Au - i(1 - \delta) (B_a)^{\text{op}} u,$$

where we used (32) in the last step. The lemma now follows. \square

Lemma 6.15. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then there exists a $\delta_0 \in (0, 1)$ such that for all $\delta \in [\delta_0, 1)$ there exists an $M > 0$ such that $D(A_\delta) \subset W^{1,2}(\mathbb{R}^d)$ and $\|u\|_{W^{1,2}} \leq M \|u\|_{D(A_\delta)}$ for all $u \in D(A_\delta)$.*

Proof. Since $D(A) \subset W^{1,2}(\mathbb{R}^d)$, there exists an $M_1 > 0$ such that $\|u\|_{W^{1,2}} \leq M_1 \|u\|_{D(A)}$ for all $u \in D(A)$ by the closed graph theorem. Similarly the inclusion $W^{1,2}(\mathbb{R}^d) \subset D((B_a)^{\text{op}})$ implies that there exists an $M_2 > 0$ which satisfies $\|u\|_{D((B_a)^{\text{op}})} \leq M_2 \|u\|_{W^{1,2}}$ for all $u \in D((B_a)^{\text{op}})$. Let $\delta_0 = (1 - \frac{1}{2M_1 M_2}) \vee \frac{1}{2}$ and $\delta \in [\delta_0, 1)$. If $u \in D(A)$ then $u \in D(A_\delta)$ by Lemma 6.14. Therefore

$$\begin{aligned} \|u\|_{W^{1,2}} &\leq M_1 (\|u\|_2 + \|Au\|_2) \leq M_1 (\|u\|_2 + \|A_\delta u\|_2 + (1 - \delta) \|(B_a)^{\text{op}} u\|_2) \\ &= M_1 \|u\|_{D(A_\delta)} + (1 - \delta) M_1 \|(B_a)^{\text{op}} u\|_2 \leq M_1 \|u\|_{D(A_\delta)} + (1 - \delta) M_1 M_2 \|u\|_{W^{1,2}} \end{aligned}$$

for all $u \in D(A)$. It follows that

$$\|u\|_{W^{1,2}} \leq \frac{M_1}{1 - (1 - \delta) M_1 M_2} \|u\|_{D(A_\delta)}$$

for all $u \in D(A)$. In particular

$$\|u\|_{W^{1,2}} \leq \frac{M_1}{1 - (1 - \delta) M_1 M_2} \|u\|_{D(A_\delta)} \tag{33}$$

for all $u \in C_c^\infty(\mathbb{R}^d)$. Note that $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ by Lemma 6.13 and the space $W^{1,2}(\mathbb{R}^d)$ is complete. Consequently (33) implies that $D(A_\delta) \subset W^{1,2}(\mathbb{R}^d)$ and

$$\|u\|_{W^{1,2}} \leq \frac{M_1}{1 - (1 - \delta) M_1 M_2} \|u\|_{D(A_\delta)}.$$

for all $u \in D(A_\delta)$ as required. \square

Lemma 6.16. *Let $u \in D(A)$. Then $\lim_{n \rightarrow \infty} A_\delta(J_n * u) = A_\delta u$ in $L_2(\mathbb{R}^d)$.*

Proof. The proof is the same as that of the ‘only if’ part of Proposition 6.11. Note that $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ by Lemma 6.13 and $D(A_\delta) \subset W^{1,2}(\mathbb{R}^d)$ by Lemma 6.15. \square

We are now in the position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\delta = \delta_0$, where δ_0 is defined as in Lemma 6.15. By Lemma 6.16 we have $\lim_{n \rightarrow \infty} A_\delta(J_n * u) = A_\delta u$ in $L_2(\mathbb{R}^d)$ for all $u \in D(A)$. Furthermore [ERS11, Proposition 2.1] gives that $\lim_{n \rightarrow \infty} (B_a)^{\text{op}}(J_n * u) = (B_a)^{\text{op}}u$ in $L_2(\mathbb{R}^d)$ for all $u \in D((B_a)^{\text{op}})$. Hence $\lim_{n \rightarrow \infty} A(J_n * u) = Au$ in $L_2(\mathbb{R}^d)$ for all $u \in D(A)$ as $A \subset A_\delta + i(1 - \delta)(B_a)^{\text{op}}$. Using Proposition 6.11 we can conclude that $C_c^\infty(\mathbb{R}^d)$ is a core for A . \square

7 Examples

In this section we present several applications of Theorems 1.2, 1.3 and 6.8 in showing the core properties for some specific degenerate elliptic operators in higher dimensions.

Example 7.1. For all $(x, y) \in \mathbb{R}^2$ let $\phi(x, y) = \frac{\pi}{4} \cos(\sin(x + y))$. Let

$$C = \begin{pmatrix} 2 \cos \phi + i \sin \phi & \sin \phi \\ -\sin \phi & 2 \cos \phi + i \sin \phi \end{pmatrix}.$$

Then $(C(x, y) \xi, \xi) \in \Sigma_{\frac{\pi}{4}}$ for all $(x, y) \in \mathbb{R}^2$ and $\xi \in \mathbb{C}^2$. Note that $B_a = 0$.

Consider the form \mathfrak{a}_0 defined by

$$\mathfrak{a}_0(u, v) = \int_{\mathbb{R}^2} (C \nabla u, \nabla v)$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^2)$. Then \mathfrak{a}_0 is closable. Let A be the operator associated with the closure of \mathfrak{a}_0 in $L_2(\mathbb{R}^2)$. Since $B_a = 0$, we can extend the contraction C_0 -semigroup S generated by $-A$ to a contraction C_0 -semigroup $S^{(p)}$ on $L_p(\mathbb{R}^2)$ for all $p \in [4 - 2\sqrt{2}, 4 + 2\sqrt{2}]$ by Proposition 1.1. Let $-A_p$ be the generator of $S^{(p)}$ for all $p \in [4 - 2\sqrt{2}, 4 + 2\sqrt{2}]$. Then the space $C_c^\infty(\mathbb{R}^2)$ is a core for A_p for all $p \in (4 - 2\sqrt{2}, 4 + 2\sqrt{2})$ by Theorem 1.2.

Example 7.2. For all $(x, y) \in \mathbb{R}^2$ let

$$C(x, y) = \begin{pmatrix} \frac{1}{\sqrt{2}}(1 + i) & e^{i(x+y)} \\ i e^{-i(x+y)} & \frac{1}{\sqrt{2}}(1 + i) \end{pmatrix}.$$

Note that

$$C = (1 + i) (\text{Re } C), \tag{34}$$

where

$$(\text{Re } C)(x, y) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\cos(x+y)+\sin(x+y)}{2} - i \frac{\cos(x+y)-\sin(x+y)}{2} \\ \frac{\cos(x+y)+\sin(x+y)}{2} + i \frac{\cos(x+y)-\sin(x+y)}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore $(C(x, y) \xi, \xi) \in \Sigma_{\frac{\pi}{4}}$ for all $(x, y) \in \mathbb{R}^2$ and $\xi \in \mathbb{C}^2$.

Consider the form \mathfrak{a}_0 defined by

$$\mathfrak{a}_0(u, v) = \int_{\mathbb{R}^2} (C \nabla u, \nabla v)$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^2)$. Then \mathfrak{a}_0 is closable. Let A be the operator associated with the closure of \mathfrak{a}_0 in $L_2(\mathbb{R}^2)$.

Using (34) and the fact that $\operatorname{Re} C$ is self-adjoint, we conclude that the space $C_c^\infty(\mathbb{R}^2)$ is a core for A by Theorem 6.8(i).

Example 7.3. Let $c_{kl} \in \mathbb{C}$ for all $k, l \in \{1, 2\}$. Suppose there exists a constant $\mu > 0$ such that

$$\operatorname{Re}(C\xi, \xi) \geq \mu \|\xi\|^2$$

for all $\xi \in \mathbb{C}^2$, where $C = (c_{kl})_{1 \leq k, l \leq 2}$. Define $A_1 = \partial_x$ and $A_2 = \cos x \partial_y + \sin x \partial_z$. Consider the form \mathfrak{a}_0 defined by

$$\mathfrak{a}_0(u, v) = \sum_{k, l=1}^2 \int_{\mathbb{R}^3} c_{kl} (A_k u) A_l v$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^3)$. Then \mathfrak{a}_0 is closable. Let A be the operator associated with the closure of \mathfrak{a}_0 in $L_2(\mathbb{R}^3)$. Then formally

$$A = - \sum_{k, l=1}^2 c_{kl} A_l A_k.$$

We have $D(A) \subset W^{1,2}(\mathbb{R}^3)$. This follows from the regularity of sub-elliptic operators on Lie groups associated to unitary representations. Specifically it follows from [ER98, Theorem 9.2.II] together with [ER94, Lemma 6.1] and [ER94, Theorem 7.2.(VI and V)] applied to the standard representation of the covering group of the Euclidean motion group (cf. [DER03, Example II.5.1]).

Hence $C_c^\infty(\mathbb{R}^3)$ is a core for A by Theorem 1.3.

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